Econ 203c, Spring 2003, Problem Set 1 Suggested Answers

1)
a) Show $cov(y_i, \varepsilon_i) < 0$

from true model:
$c_i = \beta_1 + \beta_{20}(y_i - v_i) + u_i$
$c_i = \beta_1 + \beta_{20}y_i + (u_i - \beta_{20}v_i)$
$c_i = \beta_1 + \beta_{20}y_i + \varepsilon_i$

$cov(y_i, \varepsilon_i) = cov(y_i, u_i - \beta_{20}v_i) = cov(y_i^* + v_i, u_i - \beta_{20}v_i)$
$= E[(y_i^* + v_i)(u_i - \beta_{20}v_i)] - E[y_i^* + v_i]E[u_i - \beta_{20}v_i]$

$E[(y_i^* + v_i)(u_i - \beta_{20}v_i)] = E[y_i^* u_i] + E[v_i u_i] - \beta_{20}E[v_i v_i] - \beta_{20}E[v_i y_i^*]$
$E[y_i^* u_i] = cov(y_i^*, u_i)$ since $E[u_i] = 0$
$E[v_i u_i] = cov(v_i, u_i)$ since $E[u_i] = 0$
$E[v_i y_i^*] = cov(v_i, y_i^*)$ since $E[v_i] = 0$
By assumption, $cov(y_i^*, u_i) = cov(v_i, u_i) = cov(v_i, y_i^*) = 0$.

then $E[(y_i^* + v_i)(u_i - \beta_{20}v_i)] = -\beta_{20}E[v_i v_i] = -\beta_{20}\omega^2$

$E[y_i^* + v_i]E[u_i - \beta_{20}v_i] = 0$ since $E[u_i] = 0$ and $E[v_i] = 0$.

Thus, $cov(y_i, \varepsilon_i) = -\beta_{20}\omega^2 < 0$

b) $\widehat{\beta}_2 = \dfrac{1/n\sum(c_i - \overline{c})(y_i - \overline{y})}{1/n\sum(y_i - \overline{y})^2} = \beta_{20} + \dfrac{1/n\sum(y_i - \overline{y})\varepsilon_i}{1/n\sum(y_i - \overline{y})^2}$
$plim 1/n\sum(y_i - \overline{y})\varepsilon_i = cov(y_i, \varepsilon_i)$
$plim 1/n\sum(y_i - \overline{y})^2 = var(y_i)$
Thus, $plim(\widehat{\beta}_2 - \beta_{20}) = -\beta_{20}\omega^2/var(y_i) < 0$

2)
a) This is a property of the bivariate normal distribution (see e.g. Greene).
if $u_i|v_i \sim N(\alpha + \beta v_i, \sigma_u^2(1 - \rho^2))$
where $\rho = corr(u_i, v_i)$, $\alpha = \mu_u - \beta\mu_v$, $\beta = cov(u_i, v_i)/v(v_i)$

then, $E(u_i|v_i) = \alpha + \beta v_i = 0 + E[u_i v_i]v_i = \rho v_i$

b) $\widehat{\beta}_{iv} = (w'x)^{-1}w'y$, where w is nx1.

c) $\widehat{\beta}_{iv} = (w'x)^{-1}w'(x\beta_0 + \sigma_u u) = \beta_0 + \frac{\sigma_u w'u}{w'x}$

$$\widehat{\beta}_{iv} - \beta_0 = \frac{\sigma_u w'u}{w'x} = \frac{\sigma_u w'(\rho v + \epsilon)}{w'(w\pi_0 + \sigma_v v)} = \frac{\sigma_u w'(\rho v + \epsilon)}{\pi_0 + w'\sigma_v v}$$

d) if $\sigma_v = 0$,

$E[\widehat{\beta}_{iv} - \beta_0] = E[\frac{\sigma_u w'(\rho v + \epsilon)}{\pi_0}]$

$= \sigma_u/\pi_0(\rho E[w'v] + E[w'\epsilon])$

$= 0$ since $E[w'v] = 0$ and $E[w'\epsilon] = 0$

3)

a) z_i is a valid instrument if $E[z_i u_i] = 0$ [and $cov(z_i, x_i) \neq 0$].

b) $\widehat{\beta}_{iv} = (X'ZA^{-1}Z'X)^{-1}X'ZA^{-1}Z'y$

$\widehat{\beta}_{iv} - \beta = (1/nX'ZA^{-1}1/nZ'X)^{-1}1/nX'ZA^{-1}1/nZ'u$

By WLLN,

$plim 1/nZ'X = lim 1/n \sum E[z_i x_i'] = \sum_{zx}$, assume finite

$plim 1/nZ'\epsilon = lim 1/n \sum E[z_i u_i] = 0$, by assumption

Then, $plim \widehat{\beta}_{iv} - \beta = 0$.

c) $\sqrt{n}(\widehat{\beta}_{iv} - \beta) = (1/nX'ZA^{-1}1/nZ'X)^{-1}1/nX'ZA^{-1}1/\sqrt{n}Z'u$

$plim 1/nZ'X = lim 1/n \sum E[z_i x_i'] = \sum_{zx}$, assume finite

by CLT, $1/\sqrt{n}Z'u \xrightarrow{d} N(0, \sigma_u^2 E[z_i z_i'])$

$\sqrt{n}(\widehat{\beta}_{iv} - \beta) \xrightarrow{d} N(0, \Lambda)$

$\Lambda = \sigma_u^2 \Omega E[z_i z_i'] \Omega'$

where $\Omega = (\sum_{zx}' A^{-1} \sum_{zx})^{-1} \sum_{zx}' A^{-1}$