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**Economics 203C**
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# Problem Set 5

**Due: Wednesday May 21, 2003**

**Question 1:**

1. Provide the definitions for: (a) almost sure (a.s.) convergence of the sequence of random variables $\{X_n\}$; and (b) convergence in probability of the sequence of random variables $\{X_n\}$.
2. Provide an example of a sequence of random variables$\{X_n\}$ that converges in probability but does not converge a.s.

**Question 2:**

Consider the Generalized Method of Moments (GMM) estimator defined in Lecture Note 10:

$$\widehat{\theta}_n = \arg\min_{\theta\in\Theta} m_n(\theta)' V_n^{-1} m_n(\theta),$$

where

$$\begin{aligned} V_n &\xrightarrow{p} V, \\ m_n(\theta) &= \frac{1}{n}\sum_{i=1}^{n} \varphi(y,x;\theta), \quad \text{and} \\ E_0\left[\varphi(y,x;\theta_0)\right] &= 0. \end{aligned}$$

Show that $\widehat{\theta}_n$ is a consistent estimator for $\theta_0$.
**Hint:** The proof should follow the same steps as for the proof of consistency for the MLE.

**Question 3:**

Consider the linear model given by

$$\begin{aligned} y_i &= x_i'\beta_0 + \varepsilon_i, \quad i = 1, ...n, \\ E_0\left[\varepsilon_i | x_i\right] &= 0, \end{aligned}$$

where $x_i$ is a $K \times 1$ vector of regressors, i.e., $x_i' = (x_{1i}, ..., x_{Ki})$. Let

$$z_i' = \left(x_{1i}, ..., x_{Ki}, x_{1i}^2, ..., x_{Ki}^2\right).$$

Define

$$\begin{aligned} \varphi_1(y,x;\beta) &= \left(y_i - x_i'\beta\right) x_i, \quad \text{and} \\ \varphi_2(y,x;\beta) &= \left(y_i - x_i'\beta\right) z_i. \end{aligned}$$

1. Define the population value $\beta_0$.

2. Show that when evaluated at the population value $\beta_0$, $E_0\left[\varphi_1(y,x;\beta_0)\right] = E_0\left[\varphi_2(y,x;\beta_0)\right] = 0$.

3. Define the optimal GMM estimator for $\beta_0$ based of $\varphi_1(y,x;\beta_0)$ and $\varphi_2(y,x;\beta_0)$, say $\widehat{\beta}_n^1$ and $\widehat{\beta}_n^2$.

4. Provide the asymptotic covariance matrices for $\widehat{\beta}_n^1$ and $\widehat{\beta}_n^2$ from (3).

5. Provide a consistent estimator for the asymptotic covariance matrices from (4). Show that the proposed estimators are, in fact, consistent using the assumptions put forward in Lecture Note 10.

6. Provide the asymptotic distribution for the GMM estimator based on $\varphi_2(y,x;\beta_0)$ when $V = I$.

**Question 4:**

In the Excel file **ps5q4.xls** you are provided with the data for this exercise. Note that there are six variables in this file: $y$, and $x_1, ..., x_5$, where $x_{1i} \equiv 1$ for all $i = 1, ..., n$. Also, for this exercise we use the results of question 3 above.

1. Provide the optimal estimate for $\beta_0$ from Question 3 based on $\varphi_1(y,x;\beta_0)$. Also, provide the vector of standard errors estimates for $\widehat{\beta}_n^1$ and the vector of initial estimates for $\beta_0$ based on $V = I$. Compare the point estimates and the standard error estimates for the two estimates. Discuss.

2. Repeat the exercise in (1) for $\varphi_2(y,x;\beta_0)$.

3. Compare and discuss the differences and similarities between the optimal estimates (and their corresponding vectors of standard errors) from (1) and (2).

4. Construct a consistent estimate for the optimal weight matrix base on the optimal estimate from (3) and re-estimate the optimal GMM. Discuss the results in comparison with the results obtained in (2).

5. Construct the Wald statistic for testing the hypothesis $H_0$: $\beta_2^2 + \beta_4^2 = \beta_3^2 + \beta_5^2$, bases on $\varphi_1(y,x;\beta_0)$ and $\varphi_2(y,x;\beta_0)$. What are your conclusions? Explain briefly.