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Economics 203C
Spring 2003

Problem Set 2

Due: Wednesday April 16, 2003

Question 1:

This question is based on the multivariate regression model of Lecture note 4. Chamberlain(1984). The model is as follows: for $i = 1, \ldots, N$,

$$\begin{aligned} y_{i1} &= \pi_1' x_i + u_{i1}, \\ \vdots & \qquad \vdots \\ y_{iT} &= \pi_T' x_i + u_{iT}, \end{aligned}$$

where $E(u_{it} \mid x_i) = 0$, $\mathrm{Cov}(u_{it}, u_{js}) = \sigma_{st}$ if $i = j$ and $\mathrm{Cov}(u_{it}, u_{js}) = 0$ if $i \neq j$, and

$$X = \begin{pmatrix} x_1' \\ \vdots \\ x_N' \end{pmatrix} = \begin{pmatrix} x_{11} & \cdots & x_{1K} \\ \vdots & \ddots & \vdots \\ x_{N1} & \cdots & x_{NK} \end{pmatrix}.$$

a. Assume that the relevant matrices are conformable and show the following Kronecker product properties:

1. $(A \otimes B)(C \otimes D) = AC \otimes BD$

2. $(A \otimes B)' = A' \otimes B'$

3. If A and B are non-singular, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

b. Using the Kronecker product notation, we can write the multivariate regression model as:

$$y = (I \otimes X)\pi + u, \qquad E(u \mid X) = 0, \qquad \mathrm{Var}(u \mid X) = \Sigma \otimes I_N,$$

where Σ is a $T \times T$ matrix with the s, t elements σ_{st}.

1. Show that the GLS estimator of π is given by

$$\hat{\pi} = \left(I \otimes (X'X)^{-1} X' \right) y.$$

2. Show that

$$\mathrm{Var}(\hat{\pi} \mid X) = \Sigma \otimes (X'X)^{-1}.$$

3. Suppose that there are linear restrictions on π of the form $\pi = G\theta$, where θ is an unrestricted parameter vector and G is a known matrix. Show that the GLS estimator for θ can be written as

$$\hat{\theta} = (G'A^{-1}G)^{-1} G'A^{-1}\hat{\pi},$$

where $A = \mathrm{Var}(\hat{\pi} \mid X)$.

Question 2 (question 6 in Greene, page 376):

Consider the model given by

$$\begin{aligned} y_{1i} &= \alpha_1 + \beta x_i + \varepsilon_{1i}, \\ y_{2i} &= \alpha_2 + \varepsilon_{2i}, \end{aligned}$$

for $i = 1, ..., n$. Assume that

$$\varepsilon_i = (\varepsilon_{1i}, \varepsilon_{2i})' \sim (0, \Sigma) \quad \text{and } \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}.$$

Prove that the GLS estimate applied to the system leads to the OLS estimates for α_1 and α_2, but to a mixture of two least-squares slopes from the regressions of y_1 on x, and y_2 on x. For simplicity of the algebra, assume that the mean of the x_i's is zero, i.e., $\bar{x} = 0$.

Question 3 (question 1 in Greene, page 422):

Consider the two-equation model:

$$\begin{aligned} y_{1i} &= \gamma_1 y_{2i} + \beta_{11} x_{1i} + \beta_{21} x_{2i} + \beta_{31} x_{3i} + \varepsilon_{1i}, \\ y_{2i} &= \gamma_2 y_{1i} + \beta_{12} x_{1i} + \beta_{22} x_{2i} + \beta_{32} x_{3i} + \varepsilon_{2i}, \end{aligned}$$

where

$$\varepsilon_i = (\varepsilon_{1i}, \varepsilon_{2i})' \sim (0, \Sigma) \quad \text{and } \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}.$$

a. Verify that neither equations is identified.

b. Establish whether or not the stated restrictions are sufficient for identification (or partial identification) of the model and briefly justify your answers:

1. $\beta_{21} = \beta_{32} = 0$.
2. $\beta_{12} = \beta_{22} = 0$.
3. $\gamma_1 = 0$.
4. $\gamma_1 = \gamma_2$ and $\beta_{32} = 0$.
5. $\sigma_{12} = 0$ and $\beta_{31} = 0$.
6. $\gamma_1 = 0$ and $\sigma_{12} = 0$.
7. $\beta_{21} + \beta_{22} = 1$.
8. $\sigma_{12} = 0$ and $\beta_{21} = \beta_{22} = \beta_{31} = \beta_{32} = 0$.
9. $\sigma_{12} = 0$ and $\beta_{11} = \beta_{21} = \beta_{22} = \beta_{31} = \beta_{32} = 0$.

Question 4 (question 3 in Greene, page 423):

Check the identifiability of the parameters of the following model:

$$[y_1, y_2, y_3, y_4] \begin{bmatrix} 1 & \gamma_{12} & 0 & 0 \\ \gamma_{21} & 1 & \gamma_{23} & \gamma_{24} \\ 0 & \gamma_{32} & 1 & \gamma_{34} \\ \gamma_{41} & \gamma_{42} & 0 & 1 \end{bmatrix} =$$

$$[x_1, x_2, x_3, x_4, x_5] \begin{bmatrix} 0 & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & 1 & 0 & \beta_{24} \\ \beta_{31} & \beta_{32} & \beta_{33} & 0 \\ 0 & 0 & \beta_{43} & \beta_{44} \\ 0 & \beta_{52} & 0 & 0 \end{bmatrix} + [\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4].$$