Moshe Buchinsky
Department of Economics
UCLA

Economics 203C
Spring 2003

LECTURE NOTE 2
INSTRUMENTAL VARIABLES IN PRACTICE

I. REVIEW

ENDOGENOUS REGRESSION MODEL:

$$y_i = x_i'\beta + \epsilon_i, \quad \text{but} \quad E[x_i\epsilon_i] \neq 0.$$

INSTRUMENTAL VARIABLES:

Suppose there exists z_i and $l \times 1$ vector. Let $Z = (z_1, \ldots, z_n)'$ and suppose that

$$\mathbf{1}. \qquad \operatorname*{plim}_{n\to\infty} \left(\frac{1}{n} Z'X\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} E[z_i x_i'] = \Sigma_{zx}$$

an $l \times k$ matrix with $\text{rank}(\Sigma_{zx}) = k = \dim(x_i)$; and

$$\mathbf{2}. \qquad E[z_i\epsilon_i] = 0, \quad \Longrightarrow \operatorname*{plim}_{n\to\infty} \left(\frac{1}{n} Z'\epsilon\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} E[z_i\epsilon_i] = \Sigma_{z\epsilon} = 0.$$

Then, $z_i,\ i = 1, \ldots, n$ are instrumental variables for $x_i,\ i = 1, \ldots, n$.

REMARKS:

1. z_i may include some components of x_i, lagged x_i, exogenous x_i, intercept, etc.
2. If $\text{rank}(Z'X) = \text{rank}(\Sigma_{zx}) = k$, then we must have $l \geq k$, i.e., at least as many instruments as regressors.

II. SOME SPECIAL CASES

II.1. CASE 1: $l = k$

Here $\left(\frac{1}{n}Z'X\right)$ is squared and of full rank. hence, $\left(\frac{1}{n}Z'X\right)^{-1}$ exists and for large n, also Σ_{zx}^{-1} exists.

So, the IV estimator is

$$\hat{\beta}_{IV} = (Z'X)^{-1}Z'y = (Z'X)^{-1}Z'(X\beta + \epsilon) = \beta + (Z'X)^{-1}Z'\epsilon.$$

Note that $\hat{\beta}_{GLS}$ solves

$$0 = \frac{1}{n}\sum_{i=1}^{n} z_i(y_i - x_i'\hat{\beta}_{GLS}),$$

k equations with k unknown.

II.1. CASE 2: $l > k$

We can "throw away" some instruments to make $l = k$, or we can combine them to get a better (more efficient) estimator.

As before, we have to solve

$$0 = \frac{1}{n} = \sum_{i=1}^{n} z_i(y_i - x_i'\hat{\beta}),$$

but now we have l equations with only k unknowns. So, all equations cannot hold for any $\hat{\beta}$ (but by chance).

Suppose A is an $l \times l$ matrix, with $A \xrightarrow{p} \Psi$, positive definite matrix. Then,

$$\begin{aligned} Z'y &= Z'X\hat{\beta} + Z'e \\ &= Z'X\hat{\beta}. \end{aligned}$$

$$\implies A^{-1}Z'y = A^{-1}Z'X\hat{\beta}.$$

$$\implies X'ZA^{-1}Z'y = X'ZA^{-1}Z'X\hat{\beta}.$$

$$\begin{aligned} \implies \hat{\beta} &= \left(X'ZA^{-1}Z'X\right)^{-1}X'ZA^{-1}Z'y \\ &= \left(S_{zx}'A^{-1}S_{zx}\right)^{-1}S_{zx}'A^{-1}S_{zy}, \end{aligned}$$

where

$$S_{zx} = \frac{1}{n}\sum_{i=1}^{n} z_i x_i' \quad \text{and } S_{zy} = \frac{1}{n}\sum_{i=1}^{n} z_i y_i.$$

The remaining issue to settle is how to choose A optimally.

III. HOW DO WE CHOOSE THE INSTRUMENTS?

This question can be addressed better within the context of a given model.

EXAMPLES:

1. AUTOCORRELATION ERRORS WITH LAGGED DEPENDENT VARIABLE:

$$Y_T = \alpha + \beta x_t + \gamma y_{t-1} + \epsilon_t,$$

$$\epsilon_t = \rho \epsilon_{t-1} + u_t,$$

$$u_t \sim \text{i.i.d.}(0, \sigma^2), \quad \text{independent of } x_t.$$

In this case we have

$$\begin{aligned} E[y_{t-1}\epsilon_t] &= E[y_{t-1}(\rho\epsilon_{t-1} + u_t)] \\ &= E[(\alpha + \beta x_{t-1} + \gamma y_{t-2} + \epsilon_{t-1})(\rho\epsilon_{t-1} + u_t)] \\ &= \rho\gamma E[y_{t-2}\epsilon_{t-1}] + \rho\sigma_\epsilon^2. \end{aligned}$$

If y_t and ϵ_t are stationary, then $E[y_{t-1}\epsilon_t] = E[y_{t-2}\epsilon_{t-1}]$ and

$$E[y_{t-1}\epsilon_t] = \frac{\rho\sigma_\epsilon^2}{(1-\rho\gamma)} = \frac{\rho\sigma_u^2}{(1-\rho\gamma)(1-\rho^2)},$$

since $\sigma_\epsilon^2 = \sigma_u^2/(1-\rho^2)$.

For instrument: Let $z_t = (1, x_t, x_{t-1})'$ be the instruments for $(1, x_t, y_{t-1})'$.

By assumption $E[x_s\epsilon_t] = 0, \ \forall t \neq s$. Hence,

$$\operatorname*{plim}_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} z_t\epsilon_t = E[z_t\epsilon_t] = 0.$$

So, z_t is a valid instrument. Also,

$$\operatorname*{plim}_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} z_t(1, x_t, y_{t-1}) = E[z_t(1, x_t, y_{t-1})] \equiv \Sigma_{zx}.$$

$$z_t(1, x_t, y_{t-1}) = \begin{pmatrix} 1 & x_t & y_{t-1} \\ x_t & x_t^2 & x_t y_{t-1} \\ x_{t-1} & x_t x_{t-1} & x_{t-1} y_{t-1} \end{pmatrix}.$$

$$\Longrightarrow \quad \det(\Sigma_{zx}) \neq 0.$$

REMARKS:

a. Any $x_{t-s},\ s \geq 1$ can serve as an instrument for y_{t-1}. Hence, we can take $z_t = (1, x_t, x_{t-1}, x_{t-2}, \dots)'$.

b. If x_t is a vector, i.e., $y_t = x_t'\beta + \gamma y_{t-1} + \epsilon_t$, then $\dim(x_t', x_{t-1}')' = l > \dim(x_t', y_{t-1}) = k$.

2. OMITTED VARIABLES:

$$y_i = x_i'\beta + w_i'\gamma + u_i = x_i'\beta + \epsilon_i, \quad \epsilon_i \equiv u_i - w_i'\gamma.$$

Need an instrument that is correlated with x_i, but is uncorrelated with w_i. This is a case where it is quite difficult to come up with reasonable instrument.

3. MEASUREMENT ERRORS:

A. PERMANENT INCOME:

$$y_t = \alpha + \beta z_t + u_t,$$

$$x_t = z_t + v_t,$$

but only x_t is observed. Hence,

$$y_t = \alpha + \beta x_t + (u_t - \beta v_t) = \alpha + \beta x_t + \epsilon_t, \quad \epsilon_t \equiv u_t - \beta v_t.$$

A.1. Solution 1: Use repeated measurements.

Suppose that

$$w_t = \gamma + \delta z_t + \eta_t,$$

where (u_t, v_t, η_t) are independent of z_t (e.g., w_t can be net financial wealth at time t).

Also assume that

$$E[u_t] = E[v_t] = E[\eta_t] = 0,$$

$$E[u_t v_t] = E[v_t \eta_t] = 0.$$

That is, shock to wealth is uncorrelated with transitory consumption and transitory income.

Then,

$$\begin{aligned} E[w_t x_t] &= E[(\gamma + \delta z_t + \eta_t)(z_t + v_t)] \\ &= \delta E[z_t^2] \\ &\neq 0. \end{aligned}$$

and

$$E[w_t \epsilon_t] = E[(\gamma + \delta z_t + \eta_t)(u_t - \beta v_t)] = 0.$$

Then $(1, w_t)'$ is an instrument for $(1, x_t)'$.

A.2. Solution 2: Causal model (Zellner).

Suppose that

$$z_t = \gamma + \delta w_t + \eta_t,$$

where (u_t, v_t, η_t) are independent of w_t and

$$E[u_t] = E[v_t] = E[\eta_t] = 0,$$

$$E[u_t \eta_t] = E[v_t \eta_t] = 0.$$

That is, we assume that the "permanent income" z_t is explained by w_t (financial wealth, net total wealth, education, etc.).

Then

$$E[w_t \epsilon_t] = E[w_t(u_t - \beta v_t)] = 0,$$

by assumption, and

$$E[w_t x_t] = E[w_t(\gamma + \delta w_t + \eta_t + v_t)] = \delta E[w_t^2] \neq 0.$$

Therefore a legitimate instrument for $(1, x_t)'$ is provided by $(1, w_t)'$.

REMARK: The two models relating w_t to z_t are quite different. In the first model $E[z_t\eta_t] = 0$ and $E[w_t\eta_t] \neq 0$, while in the second model the reverse is correct: $E[z_t\eta_t] \neq 0$ and $E[w_t\eta_t] = 0$. In both models, however the same instrument works.

B. RATIONAL EXPECTATION:

Model:

$$y_t = \alpha + \beta E[x_{t+1}] + u_t, \quad u_t \sim \text{i.i.d.}(0, \sigma^2)$$

and u_t is independent of x_t.

For example, y_t can be "realized investment", while x_t can be "income." We observe only (y_t, x_t), and for x_{t+1} we have

$$x_{t+1} = E_t[x_{t+1}] + v_{t+1},$$

$$E[v_{t+1} \mid x_t, x_{t-1}, \ldots, y_t, y_{t-1}, \ldots] = 0.$$

Therefore y_t can be expressed as

$$y_t = \alpha + \beta x_{t+1} + \epsilon_t, \quad \epsilon_t \equiv u_t + \beta v_{t+1}.$$

A valid instrument would then be

$$z_t = (1, x_t, x_{t-1}, x_{t-2}, \ldots, y_{t-1}, y_{t-2}, \ldots)'.$$

4. KEYNESIAN MODEL:

Consumption:

$$c_t = \alpha + \beta y_t + \epsilon_t.$$

Income equality:

$$y_t = c_t + I_t,$$

where $c_t \equiv$Consumption, $y_t \equiv$Income, and $I_t \equiv$Investment.

Here the choice of an instrument for $(1, y_t)'$ is obvious: $z_t = (1, I_t)'$.

5. SUPPLY-DEMAND MODEL:

Demand:

$$q_t = \alpha_1 + \beta p_t + \gamma_1 y_t + u_t, \quad u_t \sim \text{i.i.d.}(0, \sigma_u^2),$$

Inverse supply:

$$p_t = \alpha_2 + \phi q_t + \gamma_2 w_t + v_t, \quad v_t \sim \text{i.i.d.}(0, \sigma_v^2),$$

where $q_t \equiv$Quantity, $p_t \equiv$Price, $y_t \equiv$Income, and $w_t \equiv$Wage Rate (or Wealth).

Assume that (y_t, w_t) are exogenous. That is,

$$E[y_t u_t] = E[y_t v_t] = E[w_t u_t] = E[w_t v_t] = 0.$$

Then, $z_t = (1, y_t, w_t)'$ is uncorrelated with $(u_t, v_t)'$.

Also, if $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$, then $E[y_t q_t] \neq 0$ and $E[w_t p_t] \neq 0$. Hence, z_t is a valid instrument for

$$x_t^d = (1, p_t, y_t)'$$

in the demand equation. Similarly, z_t is a valid instrument for

$$x_s^d = (1, q_t, w_t)'$$

in the inverse supply equation.