Econ 201C: General Equilibrium and Welfare Economics

Concavity and Quasiconcavity in QLGE

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During the fifth week, I quickly went over the following proposition, and I promised I would (eventually) post my proof.

Proposition 1 *Let* $U(z,m) = v(z) + m$. *Then* U *is quasiconcave if and only if* v *is concave.*

In order to establish this proposition, recall the following lemmas.

Lemma 2 *A function* $f(x)$ *is concave if and only if* $-f(x)$ *is convex.*

Lemma 3 *A function* $f(x)$ *is convex if and only if for any constant* k, $k + f(x)$ *is convex.*

Definition 4 *The epigraph of the function* $f(x)$ *is the set* $\{(x,y) : y \geq f(x)\}$. *That is, if we were to plot* $f(x)$ *on a two-dimensional graph, the epigraph of* $f(x)$ *is the area lying above the graph of the function* $f(x)$.

Lemma 5 *A function* $f(x)$ *is convex if and only if its epigraph is a convex set.*

Definition 6 *The upper contour set of a function* $f(x)$ *at the value* k *is the set given by* $\{x : f(x) \geq k\}$.

Lemma 7 *A function* $f(x)$ *is quasiconcave if and if its upper contour sets are convex sets.*

Proof of Proposition. Recall that the upper contour set for $U(z,m)$ at k is given by

$$\{(z,m) : v(z) + m \geq k\} = \{(z,y) : y \geq k - v(z)\}.$$

This is nothing other than the epigraph for $k - v(z)$ (the area above the graph of the function $k - v(z)$).

From the lemmas, we know that $v(z)$ is concave if and only if $-v(z)$ is convex if and only if for any constant k, $k - v(z)$ is convex. We know that $k - v(z)$ is convex. This occurs if and only if the epigraph of $k - v(z)$ is a convex set, which occurs if and only if the upper contour sets for $U(z,m)$ are convex sets. This is true if and only if $U(z,m)$ is a quasiconcave function. ∎