Econ 201C: General Equilibrium and Welfare Economics

Collection of Notation

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1 Lecture 1: Monday, April 3rd, 2006

Notation 1 *p represents **prices***

Notation 2 *$Y_j \subset \mathbb{R}^\ell$ represents a **production set** for producer j. An element $y_j \in Y_j$ is said to be an input-output vector.*

Notation 3 $Y_1 + Y_2 = \{y : y = y_1 + y_2, y_1 \in Y_1, y_2 \in Y_2\}$ *- operation of set addition*

Definition 4 *A set Y is **additive** if $Y + Y = Y$.*

Definition 5 *A set Y is **convex** if for all $y, y' \in Y$, for all $\lambda \in [0,1]$, $(1-\lambda)\, y + \lambda y' \in Y$.*

Definition 6 *If a set Y is convex, additive, and $0 \in Y$, we say that Y is a **convex cone**.*

Definition 7 *A production set Y which is a convex cone is said to exhibit **constant returns to scale**.*

Notation 8 $\pi_j(p) \equiv \sup\{p \cdot y_j : y_j \in Y_j\} \equiv \sup pY_j$ *represents the maximized profits for producer j at prices p. (Note that this is in $\mathbb{R}$.)*

Notation 9 $\eta_j(p) \equiv \{y_j \in Y_j : py_j = \pi_j(p)\} = \arg\max pY_j$ *is referred to as the **supply correspondence** and represents the set of all input-output vectors which maximize profits at the prices p. (Note that this is a subset of Y_j which is in turn a subset of $\mathbb{R}^\ell$.) It could definitely be the case that the supply correspondence contains more than one such vector!*

Notation 10 $\hat{Y} = \left\{y : y = \sum_{k=1}^{\ell+1} \alpha_k y_k, y_k \in Y \text{ for all } k, \alpha_k \geq 0, \sum_{k=1}^{\ell+1} \alpha_k = 1\right\}$ *is what we refer to as the **convex hull** of Y. It is the smallest convex set containing Y.*

Notation 11 $\hat{\pi}_j(p) \equiv \sup p\hat{Y}_j$ *is the maximized profits over the convex hull of Y.*

Proposition 12 $\hat{\pi}_j(p) = \pi_j(p)$

Notation 13 *$X_i \subset \mathbb{R}^\ell$ is the **consumption set** for consumer i.*

Notation 14 *$\succeq_i$is the **preference ordering** for consumer i. We say that $x_i \succeq_i x_i'$ if x_i is **weakly preferred** to x_i'.*

Notation 15 $\gamma_i(p, w) \equiv \{x_i \in X_i : p \cdot x_i \leq w\}$ *is referred to as the **budget set** for consumer i. It is the set of all bundles in her consumption set which she can afford if we assume that she has wealth w.*

Notation 16 $\xi_i(p, w) \equiv \{x_i \in \gamma_i(p, w) : x_i \succeq_i \gamma_i(p, w)\}$, *where we define $x_i \succeq_i \gamma_i(p, w)$ if $x_i \succeq x_i'$ for all $x_i' \in \gamma_i(p, w)$, is referred to as the **demand correspondence** for consumer i. A bundle x_i is in the consumer's demand correspondence if 1) the consumer can "afford" it and 2) it is weakly preferred to all other bundles the consumer can "afford."*

2 Lecture 2: Wednesday, April 5th, 2006

Definition 17 *An* ***economy*** *is a vector* $E = \{(X_i, \succeq_i), (Y_j), \omega\}$.

Definition 18 *A* ***state of*** E *is a vector* $[(x_i), (y_j)]$.

Definition 19 *A state is* ***attainable*** *if* $\sum_{i=1}^m x_i - \sum_{j=1}^n y_j = \omega$.

Definition 20 *A* ***private ownership economy*** *is a vector* $\mathcal{E} = \{(X_i, \succeq_i), (Y_j), (\omega_i), (\theta_{ij})\}$.

Definition 21 *An attainable state is* ***Pareto-Optimal****(.com) if there is no other attainable state in which someone could be made better off without making someone worse off.*

Notation 22 $w_i(p) \equiv p \cdot \omega_i + \sum_{j=1}^n \theta_{ij}\pi_j(p)$ *is referred to as consumer* i*'s* ***wealth function*** *as a function of prices.*

Definition 23 *The* ***aggregate wealth*** *of an economy is the sum of individual wealth. That is,* $w(p) \equiv \sum_{i=1}^m w_i(p)$.

Definition 24 *A* ***price-taking equilibrium*** *for a private ownership economy* $\mathcal{E}$ *is a vector* $\left[(x_i^*), \left(y_j^*\right), p^*\right]$ *satisfying*

1. $x_i^* \in \xi_i(p^*, w_i(p^*))$
2. $y_j^* \in \eta_j(p^*)$
3. $\left[(x_i^*), \left(y_j^*\right)\right]$ *is attainable for* $\mathcal{E}$. *(i.e.* $\sum_{i=1}^m x_i^* - \sum_{j=1}^n y_j^* = \sum_{i=1}^m \omega_i$*)*

Definition 25 *Let* $\mathcal{E}$ *be a private ownership economy. The economy* $\mathcal{E}^k \equiv \underbrace{\{\mathcal{E}, \ldots, \mathcal{E}\}}_{k}$ *is referred to as the* ***k-replica*** *of the economy* $\mathcal{E}$. *It is formed by adding* $k-1$ *of each type of consumer and producer along with any endowments and profit shares held. (It is important to note here that, as defined, in the k-replica of the economy* $\mathcal{E}$*, we have that* $\sum_i \theta_{ij} = k$ *for each* j*.)*

Definition 26 *Let* $\mathcal{E}$ *be a private-ownership economy. If* $Y_j = \{0\}$ *for each* j*, we say that* $\mathcal{E}$ *is an* ***exchange economy****. (i.e. there is no production.)*

Remark 27 *Whenever we view the economy in the context of an Edgeworth box, we are implicitly assuming that there is no production. That is, we are assuming we are in the setting of an exchange economy.*

Definition 28 *We say that an economy* $\mathcal{E}$ *is* ***replica invariant*** *if* p^* *is an equilibrium price vector for* $\mathcal{E}$ *implies that* p^* *is an equilibrium price vector for* $\mathcal{E}^k$ *for all* $k \geq 1$.

Definition 29 *A state* $[(x_i), (y_j)]$ *is* ***Pareto Optimal (strongly optimal)*** *if* $[(x_i), (y_j)]$ *is attainable and there exists no attainable* $\left[(x_i'), \left(y_j'\right)\right]$ *which satisfies* $x_i \succeq_i x_i'$ *for all* i *and* $x_i \succ_i x_i'$ *for some* i.

Notation 30 *We refer to the set* $U \equiv \{(u_1(x_1), \ldots, u_n(x_n)) : [(x_i), (y_j)]$ *is attainable*$\}$ *as the* ***utility possibility set****.*

Definition 31 *A state* $[(x_i), (y_j)]$ *is* ***weakly optimal*** *if* $[(x_i), (y_j)]$ *is attainable and there exists no attainable* $\left[(x_i'), \left(y_j'\right)\right]$ *satisfying* $x_i \succ_i x_i'$ *for all* i.

Remark 32 *We will not make a big fuss about the distinction between strong optimality and weak optimality.*

Theorem 33 (First Theorem of Welfare Economics) *If* $\left[(x_i^*), \left(y_j^*\right), p^*\right]$ *is an equilibrium for* $\mathcal{E}$*, then* $\left[(x_i^*), \left(y_j^*\right)\right]$ *is (at least) weakly optimal.*

3 Lecture 3: Monday, April 10th, 2006

Definition 34 *The set* $A_i(x_i) \equiv \{x_i' \in X_i : x_i' \succeq_i x_i\}$ *is referred to as the* ***at least as good as set*** *(or* ***weak preference set****) for individual* i *at the bundle* x_i*.*

Proposition 35 *If* u_i*, the utility function representing preferences* $\succeq_i$*, is quasiconcave, then* $A_i(x_i)$ *is a convex set for all* x_i*.*

Definition 36 *The set* $B_i(x_i) \equiv \{x_i' \in X_i : x_i' \succ_i x_i\}$ *is referred to as the* ***better than set*** *(or* ***strict preference set****) for individual* i *at the bundle* x_i*.*

Definition 37 $y \in Y$ *is* ***efficient*** *if* $Y \cap \left\{y + \mathbb{R}_+^\ell \setminus \{0\}\right\} = \emptyset$*.*

Theorem 38 (Separating Hyperplane Theorem) *Let* A *and* B *be convex sets such that* $A \cap B = \emptyset$*. Then there exists some* p *such that either*

$$\sup p \cdot A \leq \inf p \cdot B$$

or

$$\sup p \cdot B \leq \inf p \cdot A$$

Proposition 39 *Define* $\pi(p) \equiv \sup p \cdot Y$*, where* $Y \equiv \sum_j Y_j$*. Then* $\pi(p) = \sum_j \pi_j(p)$*.*

4 Lecture 4: Wednesday, April 12th, 2006

Definition 40 *Let $\omega_i = \bar{\omega}_i + r_i$. We refer to r_i as the **alienable** portion of i's endowment. We refer to $\bar{\omega}_i$ as the **inalienable** portion of i's endowment.*

Remark 41 *Market socialists believed that there were some potential moral gains by giving the alienable portion of each individual's endowment to the firms (since there is "no justification" for anyone gaining from the alienable portion of his/her endowment) and then assigning to each individual a share of the total pie. In Professor Ostroy's notes, he comments that we will "take it for granted that there is something wrong with these implications." The fact that standard general equilibrium model is consistent with such a conclusion, Ostroy believes, is more a weakness of the theory than a confirmation of the validity of market socialism. This weakness is filled when we introduce **incentives** into the model, which we will do later in the course.*

Definition 42 *A **market socialist economy** is a vector*

$$\mathcal{E}^M \equiv \{(X_i, \succeq_i), (Y_j + r_j), (\bar{\omega}_i), (\theta_i)\}$$

where $\omega = \sum_i \bar{\omega}_i + \sum_j r_j$, $\sum_j r_j = \sum_i r_i$, $\theta_i \geq 0$ for all i, and $\sum_i \theta_i = 1$.

Definition 43 *The **market socialist wealth function** is defined by*

$$w_i^M(p) \equiv \theta_i w^M(p)$$

where $w^M(p) = \sum_i p\omega_i + \sum_j \pi_j(p)$

Definition 44 *A **linear programming problem** is characterized by a triple (A, b, c), where A is an $m \times n$ matrix, b is an $m \times 1$ vector, and c is an $n \times 1$ vector.*

Definition 45 *The **primal** of the linear programming problem (A, b, c) is the optimization problem*

$$\max_x c \cdot x \quad s.t.\ Ax \leq b,\ x \geq 0 \tag{P}$$

where x is an $n \times 1$ vector.

Definition 46 *The **dual** of the linear programming problem (A, b, c) is the optimization problem*

$$\min_y y \cdot b \quad s.t.\ yA \geq c,\ y \geq 0 \tag{D}$$

Definition 47 *Let $A = \begin{bmatrix} A^1 & \cdots & A^n \end{bmatrix}$. We refer to $A^j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}$ as the jth **activity vector (input-output vector)**. The sign convention is that $a_{ij} > 0$ implies that activity j uses i as an input, and $a_{ij} < 0$ implies that activity j produces i as an output.*

Definition 48 *We refer to the scalar $x_j \geq 0$ as the **activity level** for activity j. The vector $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is the **activity vector**.*

Remark 49 *Activities are linear. Suppose we choose the activity level vector $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. Then $Ax = \sum_{j=1}^n A_j x_j$. That is, Ax represents the input-output vector resulting from the choice of x_1 units of activity A_1, x_2 units of activity A_2,..., and x_n units of activity A_j.*

Remark 50 *It is natural to think of c_j as the **revenue** of operating activity j at the unit level. (And hence $c_j x_j$ as the revenue of opearting activity j at the x_j level.) The total revenue for operating activity 1 at the level x_1,..., activity n at the level x_n is therefore $c_1x_1 + \cdots + c_nx_n \equiv c \cdot x$. The primal asks us to maximize this revenue.*

5 Lecture 5: Monday, April 17th, 2006

Proposition 51 (Primal Dual Inequality) *Suppose $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D). (i.e. $A\bar{x} \leq b$, $x \geq 0$ and $\bar{y}A \geq c$, $y \geq 0$). Then, $c\bar{x} \leq \bar{y}b$.*

Proposition 52 *If $\bar{x}$ and $\bar{y}$ are feasible and $c\bar{x} = \bar{y}b$, then $\bar{x}$ is optimal for (P) and $\bar{y}$ is optimal for (D).*

Proposition 53 *If $\bar{x}$, $\bar{y}$ are feasible for (P), (D) respectively, and*

1. $\bar{y}_i > 0$, *then* $\sum_j a_{ij}\bar{x}_j = b_j$ $\left(\iff \bar{y}_i \left(\sum_j a_{ij}\bar{x}_j - b_i\right) = 0\right)$
2. $\bar{x}_j > 0$, *then* $\sum_i \bar{y}_i a_{ij} = c_j$ $\left(\iff \bar{x}_j \left(\sum_i \bar{y}_i a_{ij} - c_j\right) = 0\right)$

Proposition 54 (Fundamental Theorem of Linear Programming) *$\bar{x}$ is optimal for (P) if and only if there is an optimal solution $\bar{y}$ for (D) and $c\bar{x} = \bar{y}b$.*

Definition 55 *Let (A, b, c) be a general linear programming problem. Define*

$$h(b) = \max\{cx : Ax \leq b, x \geq 0\}$$

*To be the **optimal value function** as a function of b. (That is, $h(b)$ tells us that, if we have the resources b, our maximized revenues are $h(b)$.)*

Definition 56 *A function $h : \mathbb{R}^m \to \mathbb{R}$ is said to be **positively homogeneous** (**homogeneous of degree one**) if $h(\lambda x) = \lambda h(x)$ for all $x \in \mathbb{R}^m$, for all $\lambda > 0$.*

Proposition 57 *The value function for a linear programming problem is positively homogeneous. That is, $h(\lambda b) = \lambda h(b)$ for all $\lambda > 0$*

Proposition 58 (Euler's Theorem) *If h is positively homogeneous and differentiable, then $h(a) = \nabla h(a) \cdot a$.*

Definition 59 *A function $h : \mathbb{R}^m \to \mathbb{R}$ is **superadditive** if $h(a + a') \geq h(a) + h(a')$ for all $a, a' \in \mathbb{R}^m$.*

Definition 60 *A function $h : \mathbb{R}^m \to \mathbb{R}$ is **concave if***

$$h(\lambda a + (1 - \lambda) a') \geq \lambda h(a) + (1 - \lambda) h(a')$$

for all $\alpha, a' \in \mathbb{R}^m$ and $0 \leq \lambda \leq 1$.

Proposition 61 *If h is positively homogeneous and superadditive, then h is concave.*

Definition 62 *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is concave. Define the set*

$$\partial f(z^*) \equiv \{p \in \mathbb{R}^n : f(z^*) - p \cdot z^* \geq f(z') - p \cdot z' \ \forall z'\}$$

*to be the **subdifferential** of f and z^*.*

Proposition 63 *If $f : \mathbb{R}^n \to \mathbb{R}$ is concave, then $\partial f(z)$ is a convex set for all z.*

Proposition 64 *If $f : \mathbb{R}^n \to \mathbb{R}$ is concave and differentiable at z^*, then $\partial f(z^*) = \{p\} = \nabla f(z^*)$. That is, the subdifferential is a singleton and exactly coincides with the gradient vector.*

6 Lecture 6: Wednesday, April 19th, 2006

Proposition 65 *Suppose $h : \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous. Then h is additive along a ray. That is, if $b' = tb$, then $h(b) + h(b') = h(b + b')$.*

Proposition 66 *If $h : \mathbb{R}^n \to \mathbb{R}$ is everywhere additive and positively homogeneous, then h is linear. (That is, $\forall b, b' \in \mathbb{R}^n$ for all $\alpha \geq 0$, $h(b + \alpha b') = h(b) + \alpha h(b')$.)*

Proposition 67 *If $h : \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous and subadditive (i.e. $\forall b, b' \in \mathbb{R}^n$, $h(b + b') \leq h(b) + h(b')$), then we have that h is convex.*

Definition 68 *Define $h^*(y) = h(b) - y \cdot b$ to be the **conjugate** of h at y, where b is optimal at y. Note that it is a function of "things that are dual to b."*

Remark 69 *If we think of $h(b)$ as revenues and $-y \cdot b$ as costs, then $h^*(y)$ is analogous to maximized profits when input prices are y. (And output prices are normalized to be 1.)*

Proposition 70 *The conjugate function of a homogeneous function is either 0 or $+\infty$.*

Remark 71 *Earlier, we noted that in a constant returns to scale economy (i.e. the setting of a linear programming problem), profits (the conjugate function) must be either 0 or $+\infty$.*

Proposition 72 *If $h : \mathbb{R}^m \to \mathbb{R}$ is a concave function, then $\partial h(b)$ is a singleton almost everywhere. (This proposition is not too relevant for this class.)*

Definition 73 *Define the function*

$$\varphi_{b,b'}(t) = \frac{h(b + tb') - h(b)}{t},\ t > 0$$

*to be the **difference quotient** for h at the point b in the direction b'.*

Proposition 74 *Suppose $b' = \alpha b$. Then $\varphi_{b,b'}(t) = \alpha h(b)$ for all t.*

Definition 75 *Suppose*

$$Dh(b; b') = \lim_{t \to 0} \frac{h(b + tb') - h(b)}{t} \text{ exists.}$$

*We refer to $Dh(b; b')$ as the **directional derivative** of h at the point b in the direction b'.*

Proposition 76 *The directional derivative is positively homogeneous. That is,*

$$Dh(b; \lambda b') = \lambda Dh(b; b')$$

for all b' and for all $\lambda > 0$.

Proposition 77 *Suppose h is a concave function. Then,*

$$\varphi_{b,b'}(t) = \frac{h(b + tb') - h(b)}{t}$$

is decreasing in t.

Proposition 78 *Suppose $y \in \partial h(b)$. Then $\forall b'$, $yb' \geq Dh(b; b')$.*

Proposition 79 *If h is differentiable at b, then there is a unique $y \in \partial h(b)$ and we will therefore have $yb' = Dh(b; b')$ for all b'.*

7 Lecture 7: Monday, April 24th, 2006

Proposition 80 *Let $b' = \sum_{k=1}^{n} \alpha_k e_k$. Suppose h is differentiable at b. Then we have that*

$$Dh(b; b') = Dh\left(b; \sum_{k=1}^{n} \alpha_k e_k\right) = \sum_{k=1}^{n} \alpha_k Dh(b; e_k).$$

Remark 81 *If h is not differentiable at b, it may be the case that*

$$Dh(b; b' + b'') > Dh(b; b') + Dh(b; b'').$$

Proposition 82 *If f is concave and $p \cdot d \geq Df(z; d)$ for all d, then $p \in \partial f(z)$.*

Remark 83 *The condition $p \cdot d \geq Df(z; d)$ is analogous to the first order necessary condition for optimality. If f is a concave function, then the necessary conditions become sufficient conditions, and we therefore have that $p \in \partial f(z)$. (In particular, z is optimal)*

Remark 84 *If f is not concave, then the condition $p \cdot d \geq Df(z; d)$ says only that z is a locally optimal point. Without concavity, we do not know anything about the global properties of f and therefore cannot say whether or not z is a globally optimal point.*

7.1 Quasilinear General Equilibrium Model

Definition 85 *In a quasilinear general equilibrium (QLGE) model, the **consumption set** for consumer i is given by $X_i = \mathbb{R}_+^{\ell} \times \mathbb{R}$. A typical element is $(x, m) \in X_i$.*

Definition 86 *We say that x is a vector of **non-money commodities**. These are the standard commodities we have analyzed in most other classes.*

Definition 87 *The commodity m is known as the **money commodity**.*

Remark 88 *The name "money commodity" might be misleading in that it has the word "money" in its name. It is in fact not related at all to the money (or bonds with zero interest rate) that has shown up in macroeconomics courses. It is merely a ficticious commodity which always has a marginal utility of one and serves the purpose of being a basis of comparison. We normalize its associated price to be one.*

Axiom 89 (Money Invariance) *In the QLGE model, assume that preferences satisfy*

$$(x, m) \sim_i (x', m') \iff (x, m + \alpha) \sim_i (x', m' + \alpha) \text{ for all } \alpha \in \mathbb{R}.$$

That is, the amount of money commodity consumed has no effect on the preference ordering of the non-money commodities.

Axiom 90 (Money Monotonicity) *In the QLGE model, assume that preferences satisfy $(x, m) \succ_i (x, m')$ whenever $m > m'$. That is, the money commodity is a good.*

Proposition 91 *If preferences $\succeq_i$ are continuous, transitive, complete (standard assumptions), and they satisfy the additional axioms of money invariance and money monotonicity, they can be represented by a utility function of the form $U_i(x, m) = u_i(x) + m$.*

Proposition 92 *Suppose $\ell = 1$. Then $MRS_{x,m}$ is not a function of m. That is,*

$$MRS_{x,m} = \frac{\partial U_i / \partial x}{\partial U_i / \partial m} = \frac{u_i'(x)}{1} = u_i'(x)$$

Remark 93 *The additional assumptions of the QLGE model over the standard model are two fold*

1. *Preferences $\succeq_i$ can be represented by a utility function of the form $U_i(x, m) = u_i(x) + m$.*

2. *The consumption set for money is unbounded. That is,* $m \in \mathbb{R}$.

Remark 94 *In addition, we typically assume that the endowment for individual* i *is given by* $\Omega_i = (\omega_i, 0) \in X_i$. *In particular, we assume that each individual is endowed with zero units of the money commodity. (This does not mean that the money commodity has no role in the economy, since we do allow the consumption of the money commodity to be negative!)*

Definition 95 *Let* $Z_i \equiv \{z : z = x - \omega_i : z \geq -\omega_i\}$ *be the* ***net trade set*** *for individual* i. *Equivalently, we cay define* $Z_i = \mathbb{R}_+^\ell - \omega_i$.

Definition 96 *The* ***actual trade set*** *is given by* $Z_i \times \mathbb{R}$.

Definition 97 *Define the function* $v_i(z_i) = u_i(z_i + \omega_i)$, $z_i \in Z_i$. *Preferences in the QLGE model can therefore be represented by* $U_i(x_i, m_i) = v_i(z_i) + m_i$. *It is important to note that the function* v_i *completely characterizes an individual, since it has built into it the individual's preferences, actual trade set, and initial endowment.*

Definition 98 *A* ***quasilinear exchange economy*** *is a vector* $\{(v_i)\}$.

Definition 99 *An allocation* $[(z_i), (m_i)]$ *is* ***feasible*** *if* $\sum_{i=1}^n z_i = 0$ *and* $\sum_{i=1}^n m_i = 0$, *and* $z_i \in Z_i$ *for all* i.

Definition 100 ***Prices*** *are given by the vector* $(p, 1)$, $p \in \mathbb{R}^\ell$. *Here, we normalize the price of the money commodity to be one, and we usually suppress mention of its price altogether.*

Definition 101 $\gamma_i((p,1), w_i(p,1)) = \{(z_i, m_i) : p \cdot z_i + m_i \leq w_i(p,1)\}$ *is the* ***budget set*** *for individual* i.

Definition 102 $\xi_i((p,1), w_i(p,1)) = \{(z_i, m_i) \in \gamma_i : (z_i, m_i) \succeq_i \gamma_i\}$ *is consumer* i*'s* ***demand correspondence***.

Remark 103 *In a private ownership economy, we typically assume that* $w_i(p,1) = 0$. *Therefore, the budget sets become* $\gamma_i(p,0) = \{(z_i, m_i) : p \cdot z_i + m_i \leq 0\}$.

Definition 104 *A vector* $[(z_i), (m_i), p]$ *is a* ***price-taking equilibrium*** *if*

1. $[(z_i), (m_i)]$ *is feasible,*

2. $p \cdot z_i + m_i = 0$ *for all* i, *and*

3. $p \in \partial v_i(z_i)$ *for all* i.

8 Lecture 8: Wednesday, April 26th, 2006

Definition 105 *Define* $v_i^*(p, w_i) = \max_{z_i, m_i} \{v_i(z_i) + m_i : p \cdot z_i + m_i = w_i\}$ *to be the* ***indirect utility*** *function for individual* i *at prices* p *with exogenous "gift"* w_i. *(For the standard indirect utility function for the QLGE model, we have that* $w_i = 0$. *That is,* $v_i^*(p) = v_i^*(p, 0)$.*)*

Proposition 106 $v_i^*(p, w_i) = v_i^*(p) + w_i$.

Definition 107 *Define* $\partial v_i(z_i^*; w_i) = \{p : v_i(z_i^*) - p \cdot z_i^* + w_i \geq v_i(z_i) - p \cdot z_i + w_i \ \forall z_i\}$ *to be the* ***subdifferential*** *for individual* i *at the net trade vector* z_i^* *with exogenous "gift"* w_i. *(The standard subdifferential is* $\partial v_i(z_i^*) = \partial v_i(z_i^*; 0)$.*)*

Proposition 108

$$\begin{aligned} \partial v_i(z_i^*; w_i) &= \{p : v_i(z_i^*) - p \cdot z_i^* + w_i \geq v_i(z_i) - p \cdot z_i + w_i \ \forall z_i\} \\ &= \{p : v_i(z_i^*) - p \cdot z_i^* \geq v_i(z_i) - p \cdot z_i \ \forall z_i\} = \partial v_i(z_i^*) \end{aligned}$$

Definition 109 *Let* $[(z_i, m_i)]$, $z_i \in Z_i$ *be feasible. That is, let* $\sum_{i=1}^n \begin{bmatrix} z_i \\ m_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. $[(z_i, m_i), p]$ *is a* ***price taking equilibrium*** *if*

1. $v_i^*(p) = v_i(z_i) + m_i$ *where* $p \cdot z_i + m_i = 0$ *for all* i *OR*
2. $p \in \partial v_i(z_i, m_i) \equiv \{p : v_i(z_i) + m_i \geq v_i(z_i') + m_i' \ \forall z_i', m_i'\}$, $p \cdot z_i + m_i = 0$ *for all* i.

Definition 110 *Alternatively,* $[(z_i), p]$, $z_i \in Z_i$ *is a* ***price-taking equilibrium*** *if* $\sum_{i=1}^n z_i = 0$ *and*

1. $v_i^*(p) = v_i(z_i) - p \cdot z_i$ *for all* i *OR*
2. $p \in \partial v_i(z_i) = \{p : v_i(z_i) - p \cdot z_i \geq v_i(z_i') - p \cdot z_i' \ \forall z_i'\}$ *for all* i.

Remark 111 *In the second definition, we have basically replaced* $m_i = -p \cdot z_i$ *for each individual. Note that* $\sum_{i=1}^n m_i = 0$ *since* $\sum_{i=1}^n m_i = -p \sum_{i=1}^n z_i = -p \cdot 0 = 0$.

Definition 112 *A feasible allocation* $[(z_i, m_i)]$ *is* ***Pareto optimal*** *if*

$$\sum_{i=1}^n v_i(z_i) = \max \left\{ \sum_{i=1}^n v_i(z_i) : \sum_{i=1}^n z_i = 0, \ z_i \in Z_i \right\}$$

Remark 113 *Note that allocation of the money commodity does not enter into the definition of a Pareto optimal allocation, except with regards to feasibility. That is, if* $[(\bar{z}_i, 0)]$ *is Pareto optimal, so is* $[(\bar{z}_i, m_i)]$, *whenever* $\sum_{i=1}^n m_i = 0$.

Definition 114 *We say that* $\mathcal{E}_I^{QL}(\bar{z}) = \left\{(v_i)_{i \in I}\right\}$ *is a* ***quasilinear exchange economy with exogenous gift*** $\bar{z}$ *if* $\sum_{i \in I} z_i = \bar{z}$ *is the aggregate net trade constraint, where* $I = \{1, \ldots, n\}$ *is the index set for individuals.*

Definition 115 *We refer to the value* $v_I(\bar{z}) = \max\{\sum_{i=1}^n v_i(z_i) : \sum_{i=1}^n z_i = \bar{z}\}$ *as the* ***maximum value of the economy*** $\mathcal{E}_I^{QL}(\bar{z})$.

Remark 116 *In essence,* $v_I(\bar{z})$ *is the conjugate function for a ficticious "representative agent" whose preferences are defined to be the sum of all the individual preferences.*

Definition 117 *The value* $v_I(\bar{z}) - v_I(0)$ *is referred to as the* ***social value*** *of moving the resource constraint from* $\sum_{i=1}^n z_i = 0$ *to* $\sum_{i=1}^n z_i = \bar{z}$.

Proposition 118 *If* v_i *is concave for all* i, *then* $v_I(\bar{z})$ *is concave in* $\bar{z}$.

Proposition 119 *A price taking equilibrium* $[(z_i^*), p]$ *is Pareto optimal.*

Definition 120 *Define* $v_{-i}(\bar{z}) = v_{I \setminus \{i\}}(\bar{z}) = \max\left\{\sum_{j \neq i} v_j(z_j) : \sum_{j \neq i} z_j = \bar{z}\right\}$ *to be the* ***maximum value*** *of the economy* $\mathcal{E}_{-i}^{QL}(\bar{z})$, *which consists of all agents except for individual* i.

Definition 121 *The* ***marginal product*** *of individual* i *is defined to be* $MP_i \equiv v_I(0) - v_{-i}(0)$.

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Definition 122 *The **window operator** is defined by*

$$(v_1 \boxplus v_2)(\bar{z}) = \max\{v_1(z_1) + v_2(z_2) : z_1 + z_2 = \bar{z}\}$$

In the context of the quasilinear model, the window operator forms a sort of "aggregated individual" from the preferences of individuals 1 and 2.

Definition 123 *The **marginal social cost (benefit)** in the direction y is defined by*

$$MSC(y) = Dv_I(0; y) = \lim_{t \downarrow 0} \frac{v_I(0 + ty) - v_I(0)}{t}$$

and represents a money measure for how much the additional resources y are worth to society, at least infinitessimally.

Definition 124 *The **marginal private cost (benefit)** in the direction y is defined by*

$$MPC(y) = Dv_i(z_i; y) = \lim_{t \downarrow 0} \frac{v_i(z_i + ty) - v_i(z_i)}{t}$$

and represents a money measure for how much the additional resources y are worth to individual i, at least infinitessimally.

Proposition 125 *When v_I and v_i are differentiable, we have that $\nabla v_I(0) = \nabla v_i(z_i) = p$. Further, we have that*

$$Dv_I(0; y) = py = Dv_i(z_i, y)$$

That is, marginal social and private incentives are exactly aligned.

Definition 126 *The **marginal product** of individual i is given by*

$$MP_i = v_I(0) - v_{-i}(0)$$

Definition 127 *The **total social cost (benefit)** of the trade $-z$ is given by*

$$\Delta v_{-i}(-z) = v_{-i}(-z) - v_{-i}(0)$$

and represents a money measure for the value of the trade $-z$ to individuals $-i$.

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Proposition 128 *In general, for a price-taking equilibrium,* $\sum_{i=1}^{n} MP_i \geq v_i^* (p)$ *for all* i*.*

10.1 Flattening Effect of Large Numbers

Definition 129 *The* ***total cost (benefit)*** *of the trade* z *is given by*

$$\Delta v_I (z) = v_I (z) - v_I (0)$$

and represents the maximum a group of individuals I *would be willing to pay (or would need to be compensated) for the trade* z*.*

Let $\Delta v_{2I} (z) = [\Delta v_I \boxplus \Delta v_I] (z) = \max_{z_1, z_2} \{\Delta v_I (z_1) + \Delta v_I (z_2) : z_1 + z_2 = z\}$ be the maximum amount of money two identical groups I of individuals would be willing jointly to pay for the trade z.

Proposition 130 $\Delta v_{kI} (z) = k \Delta v_I \left(\frac{z}{k}\right)$ *if* v *is a concave function.*

Proposition 131 *In the limit as the number of individuals goes to infinity, the social benefit function is linear. That is,*

$$\lim_{k \to \infty} \Delta v_{kI} (z) = \lim_{k \to \infty} k \Delta v_I \left(\frac{z}{k}\right) = \lim_{k \to \infty} \frac{v_I \left(\frac{1}{k} z\right) - v_I (0)}{\frac{1}{k}} = D v_I (0; z) .$$

We refer to this as the flattening effect of large numbers.

10.2 Producers

Definition 132 *The preferences for a* ***producer*** *in a quasilinear general equilibrium model are given by* $v (z) = -c (z)$ *where* $c (z)$ *is the money input to produce the trade vector* z*.*

Remark 133 *If* $c (z) = \begin{cases} 0 & z \in Y \\ \infty & z \notin Y \end{cases}$*, then we say that money is neither an input nor an output for the producer with production set* Y*.*

11 Lecture 11: Monday, May 8th, 2006

11.1 Convexifying Effect of Large Numbers

Definition 134 *Suppose* $v : \mathbb{R}^\ell \to \mathbb{R}$. *Define*

$$\hat{v}(\bar{z}) = \max\left\{\sum_{k=1}^{\ell+1} \lambda_k v(z_k) : \sum_{k=1}^{\ell+1} \lambda_k = 1, \lambda_k \geq 0, \sum_{k=1}^{\ell+1} \lambda_k z_k = \bar{z}\right\}$$

to be the ***concavified*** v. *Sometimes we write* $\hat{v} = conc(v)$. *By construction,* $\hat{v}$ *is the smallest concave function which lies above* v.

Proposition 135 *If* v *is concave, then* $v(z) = \hat{v}(z)$ *for all* z. *If* v *is not concave, then we have that* $v(z) \leq \hat{v}(z)$ *for all* z *and for some* z' *it may be that* $v(z) < \hat{v}(z')$.

In nothing but a change of notation (towards more general notation), let

$$h(1, \bar{z}) = \max\left\{\sum_z x_z v(z) : x_z \geq 0, \sum_z x_z = 1, \sum x_z z = \bar{z}\right\}.$$

It can be shown that $h(1, \bar{z}) = \hat{v}(\bar{z})$.

Proposition 136 *Let* $k > 0$ *and* v *be concave. Then* $h(k, k\bar{z}) = kh(1, \bar{z})$, *where*

$$h(k, k\bar{z}) = \max\left\{\sum_z x_z v(z) : x_z \geq 0, \sum_z x_z = k, \sum_z x_z z = k\bar{z}\right\}.$$

Definition 137 *Suppose we make the restriction* $x_z \in \{0, 1, 2, \dots\}$. *Then define*

$$h^{\#}(1, \bar{z}) = \max\left\{\sum_z x_z v(z) : x_z \in \{0, 1, 2, \dots\}, \sum_z x_z = 1, \sum_z x_z z = \bar{z}\right\}$$

to be the ***integer constrained concavified*** v.

Proposition 138 *If* v *is concave, then* $h^{\#}(1, \bar{z}) = h(1, \bar{z})$. *That is, if* v *is a concave function, then the integer constrained concavified* v *exactly coincides with the concavified* v.

Proposition 139 *If* v *is not concave, then we may have that* $h^{\#}(1, \bar{z}) < h(1, \bar{z})$.

Proposition 140 *For any* $k \geq 1$, *for any* v, $h^{\#}(k, k\bar{z}) \geq kh^{\#}(1, \bar{z})$. *(Or* $\frac{1}{k}h^{\#}(k, k\bar{z}) \geq h^{\#}(1, \bar{z})$ *equivalently.)*

Proposition 141 (Convexifying Effect of Large Numbers) *If* v *does not have unbounded nonconcavities, then*

$$\lim_{k\to\infty} \frac{1}{k} h^{\#}(k, k\bar{z}) = h(1, \bar{z}) = \hat{v}(\bar{z}).$$

Remark 142 *If* v *has unbounded nonconcavities, (that is, if there exists an unbounded set on which* v *is convex) then we may have that* $\frac{1}{k}h^{\#}(k, k\bar{z}) > h^{\#}(1, \bar{z})$ *and*

$$\lim_{k\to\infty} \frac{1}{k} h^{\#}(k, k\bar{z}) = +\infty.$$

That is, the convexifying effect of large numbers breaks down in the face of unbounded nonconcavities.

Definition 143 *If* $h^{\#}(k, k\bar{z}) = h(k, k\bar{z})$ *(i.e., the optimal solution for* h *is in the integers), we say that the economy is* ***integrally optimal***.

Definition 144 *If* $h(k, k\bar{z}) = kh(1, \bar{z})$ *(i.e.,* h *is positively homogeneous), we say that the economy is* ***replica invariant***.

Proposition 145 *An economy is* ***integrally optimal*** *if and only if it is* ***replica invariant***.

Proposition 146 $U(z, m) = v(z) + m$ *is quasiconcave (i.e., has convex upper contour sets) if and only if* v *is concave.*

11.2 Assignment Model

Definition 147 *An* ***assignment model*** *is a triple* (B, S, V) *of buyers* $B = \{1, \ldots, n\}$, *sellers* $S = \{1, \ldots, m\}$, *and values of matches* $V(b, s) \geq 0$.

Definition 148 *An* ***assignment*** *is a sequence of values* $\{x(b, s)\}_{b,s}$ *taking on either* 0 *or* 1, *where* $x(b, s) = 1$ *if* b *is matched with* s *and* $x(b, s) = 0$ *otherwise.*

Definition 149 *An assignment* $\{x(b, s)\}_{b,s}$ *is* ***feasible*** *if*

1. $\sum_b x(b, s) \leq 1$ *for all* s. *(i.e., each seller can have at most one match)*
2. $\sum_s x(b, s) \leq 1$ *for all* b. *(i.e., each buyer can have at most one match)*
3. $x(b, s) \in \{0, 1\}$ *for all* s *and* b. *(i.e. integer constrained assignments)*

Definition 150 *An assignment model* (B, S, V) *is a* ***double auction model*** *if the matrix of values* $V(b, s) = \max\{b - s, 0\}$. *(We sometimes say that such a model is a commodity representation.)*

Notation 151 *Let* $y_b(b, s)$ *denote the amount buyer* b *receives in the match* (b, s). *Let* $y_s(b, s)$ *denote the amount seller* s *receives in the match* (b, s).

Definition 152 *A vector* $[\{y_b(b, s)\}, \{y_s(b, s)\}, \{x(b, s)\}]$ *of payments and an assignment is said to be an* ***equilibrium*** *if*

1. $\{x(b, s)\}$ *is feasible*
2. $y_b(b, s) + y_s(b, s) \geq V(b, s)$ *for all* b *and* s. *(no "better" matches)*
3. $x(b, s) = 1 \Rightarrow y_b(b, s) + y_s(b, s) = V(b, s)$. *(feasible split)*

12 Lecture 12: Wednesday, May 10th, 2006

12.1 More Convexifying Effect of Large Numbers

Remark 153 *It follows from the definition that* $h^{\#}(k, k\bar{z}) \geq h^{\#}(k-1,(k-1)\bar{z})$. *In fact, as we let* $k \to \infty$, *the amount of resources in the economy becomes unbounded. (i.e.* $k\bar{z} \to \infty$*) Can we still say anything intelligent about the economy as the number of replications goes to infinity? For this, we "rescale" the economy.*

$$
\begin{aligned}
\frac{1}{k}h^{\#}(k, k\bar{z}) &= \frac{1}{k}\max\left\{\sum_z x_z v(z) : x_z \in \{0,1,2,\dots\}, \sum_z x_z = k, \sum_z x_z z = k\bar{z}\right\} \\
&= \max\left\{\sum_z \frac{x_z}{k} v(z) : \frac{x_z}{k} \in \left\{0, \frac{1}{k}, \frac{2}{k}, \dots\right\}, \sum_z \frac{x_z}{k} = 1, \sum_z \frac{x_z}{k} z = \bar{z}\right\} \\
&= \max\left\{\sum_z \lambda_z v(z) : \lambda_z \in \left\{0, \frac{1}{k}, \frac{2}{k}, \dots\right\}, \sum_z \lambda_z = 1, \sum_z \lambda_z z = \bar{z}\right\}
\end{aligned}
$$

It is as if the entire economy contains individuals with a total mass of 1. Here, we assign fractions λ_z *of the population to consume the consumption vector* z. *As* $k \to \infty$, *we can let* λ_z *be any real number. This gives rise to the convexifying effect of large numbers.*

Proposition 154 (Convexifying Effect of Large Numbers) $\lim_{k\to\infty} \frac{1}{k} h^{\#}(k, k\bar{z}) = \hat{v}(\bar{z})$.

12.2 More Assignment Model

Remark 155 *The assignment model can be expressed as a linear programming problem.*

Example 156 *The primal for an assignment model* (B, S, V) *is*

$$\max_{\{x(b,s)\}_{s,b}} \sum_{s\in S}\sum_{b\in B} V(b,s)\, x(b,s)$$

$$
\begin{aligned}
s.t. \sum_{s\in S} x(b,s) &\leq 1 \text{ for all } b \in B \\
\sum_{b\in B} x(b,s) &\leq 1 \text{ for all } s \in S \\
x(b,s) &\geq 0 \text{ for all } b \in B \text{ and } s \in S
\end{aligned}
$$

The dual for this model is

$$\min_{\{y_b, y_s\}_{s,b}} \sum_{b\in B} y_b + \sum_{s\in S} y_s$$

$$
\begin{aligned}
s.t.\ y_b + y_s &\geq V(b,s) \text{ for all } b \in B \text{ and } s \in S \\
y_b, y_s &\geq 0 \text{ for all } b \in B \text{ and } s \in S
\end{aligned}
$$

Remark 157 *By complementary slackness, it is necessarily the case that when* $x(b,s) > 0$, $y_b + y_s = V(b,s)$.

Remark 158 *In the assignment model, the marginal product of an infinitessimal individual, scaling up the size of the individual, is equal to the marginal product of a discrete individual.*

13 Lecture 13: Monday, May 15th, 2006

Proposition 159 *Typically, we have that* $v_i^*(p) \le MP_i$ *for all* i*, where* p *is the price vector for a price-taking equilibrium.*

Definition 160 *Let* MP_i^k *denote the* ***marginal product for individual i in the k-replica*** *of the economy.*

Proposition 161 $MP_i^k \ge MP_i^{k+1} \ge \dots$ *and* $MP_i^k \to v_i^*(p)$ *in general. This is another version of the flattening effect of large numbers.*

Remark 162 *In the simple assignment model with one buyer and one seller, we will have that* $MP_i^k = MP_i^{k+1}$ *for all* k*. That is, in this case, the flattening effect of large numbers does not work since there is perfect complementarity between buyers and sellers.*

Proposition 163 *Perfect competition is a necessary condition for self-interested behavior to lead to efficiency.*

Remark 164 *There are some counterexamples to this. Namely, when the preferences of the individuals are not opposed to one another, competition is not necessary. We claim that this is a boring case and will rule it out in all that follows.*

Proposition 165 *In general, we have that the set of Nash equilibrium strategy profiles do not intersect the set of Pareto optimal strategies. That is, Nash equilibrium strategy profiles are generally not efficient.*

Proposition 166 *A necessary condition for a Nash equilibrium strategy profile to be Pareto optimal is full appropriation.*

Remark 167 *It will turn out that this is not sufficient. We will see later in the course what the conditions are sufficient for the optimality of Nash equilibria.*

13.1 Occupational Choice Model

Definition 168 *Let* V_i *denote the set of* ***occupations*** *available for individual* i*.* $v_i \in V_i$ *is a choice of a particular occupation. Let* $\mathbf{v} = (v_i)_{i \in I}$ *denote an* ***occupation profile****.*

Definition 169 *Let* $Y_i(v_i)$ *denote the set of feasible* ***input-output*** *vectors when individual* i *chooses occupation* v_i*.*

Definition 170 *Let* $L = \{1, \dots, \ell\}$ *denote the* ***superset of commodities****. That is,* L *is the set of all possible commodities that could be produced in this economy for any choice of occupation. Let* $\mathbb{R}^\ell$ *denote the* ***commodity space****.*

Definition 171 *Let* $L(\mathbf{v})$ *denote the* ***active subset of commodities*** *under the occupation profile* $\mathbf{v}$*. That is,* $L(\mathbf{v}) \equiv \{c : z_{ci} < 0 \text{ is feasible for some } y_i \in Y_i(v_i) \text{ for some } i \in I\}$*. It is the set of commodities which can be produced under the occupation profile* $\mathbf{v}$*. We denote the* ***active commodity subspace*** *by* $\mathbb{R}^{L(\mathbf{v})} = \left\{z \in \mathbb{R}^\ell : z_c = 0 \text{ for all } c \notin L(\mathbf{v})\right\}$*.*

Definition 172 *The* ***utility function over net trades*** *is given by*

$$v_i(z_i) = \max_{y_i \in Y_i(v_i) \cap \mathbb{R}^{L(\mathbf{v})}} U_i(\omega_i + y_i + z_i | v_i)$$

where ω_i *is individual* i*'s initial endowment.*

Remark 173 *Given* $\mathbf{v}$*, we can acquire equilibrium prices* $p(\mathbf{v})$ *from which we can obtain equilibrium payoffs* $v_i^*(p(\mathbf{v}))$*. We denote* $v_i^*(p(\mathbf{v})) \equiv \pi_i(\mathbf{v})$ *as individual* i*'s payoff under the occupation profile* $\mathbf{v}$*.*

Definition 174 *An **occupational equilibrium** is an occupation profile* $\mathbf{v}$ *satisfying*

$$\pi_i(\mathbf{v}) \geq \pi_i(\mathbf{v}_{-i}, v_i') \text{ for all } v_i' \in V_i \text{ for all } i.$$

Definition 175 *Let* $g(\mathbf{v}) \equiv \max_{(z_i)} \left\{ \sum_{i \in I} v_i(z_i) : z_i \in \mathbb{R}^{L(\mathbf{v})}, \sum_{i \in I} z_i = 0 \right\}$ *be the **maximum gains from trade** under the occupational profile* $\mathbf{v}$*.*

Definition 176 *An occupational profile* $\mathbf{v}$ *is **Pareto optimal** if*

$$\mathbf{v} = \max_{v \in V_1 \times \cdots \times V_n} g(\mathbf{v})$$

Remark 177 *In general, we will have that an occupational equilibrium is NOT Pareto optimal.*

14 Lecture 14: Monday, May 17th, 2006

Definition 178 *Let* $\Delta\pi_i(\mathbf{v}; v_i') \equiv \pi_i(\mathbf{v}_{-i}, v_i') - \pi_i(\mathbf{v})$. *We refer to this quantity as the* ***private benefit*** *to* i *of switching from* v_i *to* v_i'.

Definition 179 *Let* $\Delta g(\mathbf{v}; v_i') \equiv g(\mathbf{v}_{-i}, v_i') - g(\mathbf{v})$. *We refer to this quantity as the* ***social benefit*** *of individual* i *switching from* v_i *to* v_i'.

Remark 180 *If* $\Delta\pi_i(\mathbf{v}; v_i') \cdot \Delta g(\mathbf{v}; v_i') < 0$, *we say that private and social incentives are not "properly" aligned. That is, private benefit goes in the opposite direction as social benefit. In order to have.*

Proposition 181 *Let* $\mathbf{v}$ *be an occupational equilibrium. Then* $\Delta\pi_i(\mathbf{v}; v_i')\,\Delta g(\mathbf{v}; v_i') \geq 0$ *for all* i *for all* v_i' *is a necessary condition for* $\mathbf{v}$ *to be Pareto optimal.*

Definition 182 *If* $p(\mathbf{v}_{-i}, v_i') = p(v)$ *for all* $v_i' \in V_i$, *then we say that* i *is a* ***perfect competitor***.

Definition 183 *If* $\forall\mathbf{v}$, $\forall i$, $\forall v_i' \in V_i$, $\Delta\pi_i(\mathbf{v}; v_i') = \Delta g(\mathbf{v}; v_i')$, *then we have* ***full appropriation***.

Remark 184 *Note that if we have full appropriation, then the necessary conditions for occupational equilibria being Pareto optimal are satisfied. Full appropriation is not a sufficient condition, though. We need to rule out complementarities, which we will do in the next lecture.*

15 Lecture 15: Monday, May 22nd, 2006

Proposition 185 *If a game exhibits full appropriation, and* $\mathbf{v}$ *is an occupational equilibrium, then* $\Delta g\left(\mathbf{v}; v_i'\right) \leq 0$ *for all* $v_i' \in V_i$ *for all* i.

Notation 186 *Define* $\Delta g\left(\mathbf{v}; \mathbf{v}'\right) \equiv g\left(\mathbf{v}'\right) - g\left(\mathbf{v}\right)$ *where* $\mathbf{v}' = \left(v_1', \ldots, v_n'\right)$

Definition 187 $\mathbf{v}$ *is* ***efficient*** *if* $\Delta g\left(\mathbf{v}; \mathbf{v}'\right) \leq 0$ *for all* $\mathbf{v}' \in V$.

Definition 188 *If* $\sum_{i=1}^{n} \Delta g\left(\mathbf{v}; v_i'\right) \geq \Delta g\left(\mathbf{v}; \mathbf{v}'\right)$ *for all* $\mathbf{v}, \mathbf{v}' \in V$, *then we say that the game has* ***no complementarities*** *(NC).*

Proposition 189 *Suppose a game exhibits full appropriation and no complementarities. Let* $\mathbf{v}$ *be an occupational equilibrium. Then* $\mathbf{v}$ *is efficient.*

Definition 190 *A* ***real externality*** *is the direct impact of individual* i*'s actions on individual* j*'s payoff,* $j \neq i$, *that does not go through the price system.*

Definition 191 *A* ***pecuniary externality*** *is the indirect impact of individual* i*'s actions on individual* j*'s payoff,* $j \neq i$, *through the price system.*

Notation 192 *Define* $\Delta p_c\left(\mathbf{v}; v_i'\right) \equiv p_c\left(\mathbf{v}_{-i}; v_i'\right) - p_c\left(\mathbf{v}\right)$ *be the change in the price of commodity* c *resulting from the change in occupational profile.*

Definition 193 *A game exhibits* ***perfectly elastic demand and supply*** *(PEDS) if it is the case that* $\Delta p_c\left(\mathbf{v}; v_i'\right) = 0$ *for all* $c \in L\left(\mathbf{v}\right)$ *for all* $\mathbf{v} \in V$, *for all* $v_i' \in V_i$ *and for all* i. *(That is, no one has the capacity to affect prices.)*

16 Lecture 16: Wednesday, May 24th, 2006

Definition 194 *Individual i is a **price-maker** if there exists some $v_i, v_i' \in V_i$ with $v_i \neq v_i'$ such that $p(\mathbf{v}_{-i}, v_i) \neq p(\mathbf{v}_{-i}, v_i')$ for some $\mathbf{v}_{-i}$. (i.e. individual i can affect prices)*

Remark 195 *If an economy contains at least one price-maker, we will probably have that $\Delta\pi_i(\mathbf{v}; v_i') \cdot \Delta g(\mathbf{v}; v_i') < 0$ and we will likely have that occupational equilibria will be inefficient.*

Remark 196 *The sources of the coordination problem are 1) failure of differentiability or 2) failure of concavity.*

Definition 197 *A public good z is **non-rivalrous** in the sense that the consumption of good z by individual i does not affect the capacity for individual j to consume good z.*

17 Lecture 17: Wednesday, May 31st, 2006

Definition 198 *A **Lindahl Equilibrium** for the economy* $(v_0, v_1, \ldots, v_n)$ *is a vector of individualized prices and quantities* $\left[\left(p^i\right), p^0, (z_i), z_0\right]$ *satisfying*

1. $z_i - z_0 \leq 0$ *for all* i *(quantity clearing)*
2. $\sum_{i=1}^n p^i - p^0 = 0$ *(price clearing)*
3. $v_i^*\left(p^i\right) = v_i(z_i) - p^i z_i$ *for all* i *(consumer optimality)*
4. $v_0^*\left(p^0\right) = p^0 z_0 - c_0(z_0)$*, where* $v_0(z_0) = -c_0(z_0)$*. (producer optimality)*

Definition 199 *An allocation of public goods* z *is **efficient** if*

$$z = \arg\max \left\{ \left[\sum_{i=1}^n v_i\right](z) - c_0(z) \right\}$$

Proposition 200 *Let* $\left[\left(p^i\right), p^0, (z_i), z_0\right]$ *be a Lindahl Equilibrium. Then* $z_0 \,(= z_i \;\forall i)$ *is efficient.*

Remark 201 *By the previous proposition, the idea of Lindahl pricing seems to be a great fix for the problem of public goods (and for externalities). Unfortunately, in the same sense that the assumption of price-taking behavior is naive, so is the idea that individuals would take Lindahl prices as given. Truth-telling (i.e. correctly reporting one's marginal utility) is not incentive compatible in either the situation of public goods nor of private goods.*

18 Lectures 18 and 19: Monday/Wednesday, June 5/7th, 2006

Remark 202 *For the summaries of lectures 18 and 19, I will follow rather closely the notation and definitions introduced in the notes on non-manipulable mechanisms, because I feel they are very consistent and well-organized.*

Definition 203 *An* ***institution (indirect mechanism)*** *is a vector* $\mathcal{I} = (A_1, \ldots, A_n; f)$ *of action sets* A_i *for individual* i *and a function* $f : A_1 \times \cdots \times A_n \to \mathbb{R}^{\ell+1} \times \cdots \times \mathbb{R}^{\ell+1}$ *which maps actions to outcomes.*

Definition 204 *Let* $V = V_1 \times \cdots \times V_n$ *be the set of all possible* n ***person economies****. An element* $V \ni \mathbf{v} = (v_1, \ldots, v_n)$ *is an* ***economy****.*

Definition 205 *A* ***mechanism (direct mechanism)*** *is a mapping* $F : V \to \mathbb{R}^{\ell+1} \times \cdots \times \mathbb{R}^{\ell+1}$ *in which we have that*

$$F(\mathbf{v}) = (F_1(\mathbf{v}), \ldots, F_n(\mathbf{v})) = ((z_1(\mathbf{v}), m_1(\mathbf{v})), \ldots, (z_n(\mathbf{v}), m_n(\mathbf{v}))).$$

$F_i(\mathbf{v}) = (z_i(\mathbf{v}), m_i(\mathbf{v}))$ *is the allocation of the non-money commodity and money commodities for each of the* n *individuals when the reported characteristics are* $\mathbf{v}$*.*

Definition 206 *The mechanism* F *is* ***feasible in the non-money commodities*** *if for all* $\mathbf{v} \in V$*, we have that* $(z_i(\mathbf{v})) \in Z$ *where*

- $Z = \{(z_i) : \sum_{i=1}^n z_i = 0\}$ *if the non-money commodities are private goods*
- $Z = \{(z_i) : z_i = z \ \forall i\}$ *if the non-money commodities are public goods*

Definition 207 *The mechanism* F *is* ***budget balancing*** *if* $\sum_{i=1}^n m_i(\mathbf{v}) = 0$ *for all* $\mathbf{v} \in V$*.*

Definition 208 *The mechanism is* ***efficient in the non-money commodities*** *if for all* $\mathbf{v} \in V$*,* $(z_i(\mathbf{v})) = \arg\max g(\mathbf{v})$*.*

Definition 209 *Let* $\mathbf{v} = (v_1, \ldots, v_n)$*. Define the* ***replacement*** $\mathbf{v}|\, v_i' = (v_1, \ldots, v_{i-1}, v_i', v_i, \ldots, v_n)$ *to be the vector in which the* i^{th} *element is replaced by* v_i'*. Clearly,* $\mathbf{v}|\, v_i = \mathbf{v}$*.*

Definition 210 *The* ***utility consequences*** *to individual* i *of type* v_i *who chooses announcement* v_i' *under the mechanism* F *are given by*

$$\pi_i^F(\mathbf{v}|\, v_i'; v_i) \equiv u_i(F_i(\mathbf{v}|\, v_i'); v_i) = v_i(z_i(\mathbf{v}|\, v_i')) + m_i(\mathbf{v}|\, v_i').$$

Definition 211 *We say that the mechanism* F *is* ***non-manipulable*** *(or that truth telling is a dominant strategy) if for all* i *for all* $\mathbf{v} \in V$*,*

$$\pi_i^F(\mathbf{v}) = \max_{v_i'} \pi_i^F(\mathbf{v}|\, v_i'; v_i).$$

Definition 212 *The* ***actual gains*** *at* $\mathbf{v}|\, v_i'$ *when* i *reports* v_i' *when his/her true characteristics are* v_i *are given by*

$$g(\mathbf{v}|\, v_i'; v_i) \equiv \sum_{j \neq i} v_j(z_j(\mathbf{v}|\, v_i')) + v_i(z_i(\mathbf{v}|\, v_i')).$$

Proposition 213 *Assume that* F *is efficient in the non-money commodities. Let* $\mathbf{v} = (v_1, \ldots, v_n)$ *be the true vector of characteristics in an economy. Then we have that* $g(\mathbf{v}) \geq g(\mathbf{v}|\, v_i'; v_i)$ *for all* v_i'*.*

Definition 214 *We say that individual* i *is a* ***dictatorial appropriator*** *of gains under mechanism* F *if* $\pi_i^F(\mathbf{v}|\, v_i'; v_i) = g(\mathbf{v}|\, v_i'; v_i)$*.*

Proposition 215 *If each individual* $i = 1, \ldots, n$ *is a dictatorial appropriator under mechanism* F*, then* F *is non-manipulable.*

Remark 216 *This is exactly the situation discussed in the Landsburg reading I posted on the TA website.*

Definition 217 *Define the* ***utility rights*** *of* $-i$*, or* $h_i(\mathbf{v}_{-i})$ *as the minimum amount of utility which individual* i *must ensure that individuals* $-i$ *receive.*

Definition 218 *We say that a mechanism* F ***rewards marginal products relative to utility rights*** $h = (h_1(\mathbf{v}_{-1}), \ldots, h_n(\mathbf{v}_{-n}))$ *if for all* i *and* $\mathbf{v}|\, v_i'$*,*

$$\pi_i^F(\mathbf{v}|\, v_i'; v_i) = g(\mathbf{v}|\, v_i') - h_i(\mathbf{v}_{-i}) \equiv MP_i^h(\mathbf{v}|\, v_i')$$

Proposition 219 *If* F *rewards marginal products relative to utility rights* h *for, then* F *is non-manipulable.*

Proposition 220 *It is impossible in general for a mechanism* F *to be non-manipulable, budget balancing, and efficient with respect to non-money commodities.*