Econ 201C: General Equilibrium and Welfare Economics

Mathematical Preliminaries

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1 Convexity

1.1 Convex Sets

Formally, we define a convex set as follows.

Definition 1 *A set A is **convex** if $\forall a, b \in A$, $\forall \lambda \in [0,1]$, $(1-\lambda) a + \lambda b \in A$.*

Intuitively, a set is convex if, whenever you take any two points in it, the line segment connecting these two points is completely contained within it.

Examples:

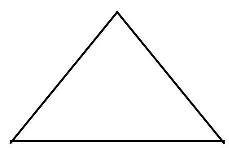

Figure 1: Convex set in two dimensions.

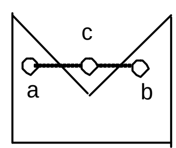

Figure 2: Non-convex set in two dimensions.

Note that the set in Figure 2 is not convex because we can find an $a \in A$ and a $b \in A$ as well as a $\lambda \in [0,1]$ such that $c = (1-\lambda) a + \lambda b \notin A$.

1.2 Concave Functions

Definition 2 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be **concave** if for any $x_0, x_1 \in Dom(f)$ and for any $\lambda \in [0,1]$, we have the following inequality*

$$f((1-\lambda) x_0 + \lambda x_1) \geq (1-\lambda) f(x_0) + \lambda f(x_1)$$

To connect the idea of a concave function with that of a convex set, we have the following proposition, which I will not prove here.

Proposition 3 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is concave if and only if the set $\{(x,y) : y \leq f(x)\} \subset \mathbb{R}^n \times \mathbb{R}$ is convex.*

Definition 4 *We refer to the set $\{(x,y) : y \leq f(x)\}$ as the **hypograph** of the function f.*

Intuitively, the hypograph of f is "the area under the curve." A function is concave if this area is a convex set.

1.3 Quasiconcave Functions

Definition 5 *A function $f:\mathbb{R}^n\to\mathbb{R}$ is said to be* ***quasiconcave*** *if for all $x_0, x_1 \in Dom(f)$ and for all $\lambda\in[0,1]$, we have the following inequality*

$$f\left((1-\lambda)x_0+\lambda x_1\right)\geq \min\left\{f(x_0), f(x_1)\right\}$$

We can connect this definition to the idea of convex sets as well with the following proposition (whose proof I will omit).

Proposition 6 *A function $f:\mathbb{R}^n\to\mathbb{R}$ is quasiconcave if and only if the set $\{x:f(x)\geq a\}\subset\mathbb{R}^n$ is convex for all $a\in\mathbb{R}$.*

Definition 7 *We refer to the set $\{x:f(x)\geq a\}\subset\mathbb{R}^n$ as the* ***upper contour set*** *for the function f at the value a.*

2 Supremum

2.1 Supremum of a Set

Definition 8 *Let $A\subset\mathbb{R}$. We say that b is an* ***upper bound*** *for A if for all $a\in A$, $a\leq b$.*

Definition 9 *A point b is said to be a* ***maximal point*** *for A if $b\in A$ and for all $a\in A$, $a\leq b$.*

We write $b=\max_{a\in A}a\equiv\max A$.

The critical distinction between an upper bound and a maximal point of a set is that the definition of the maximal point calls for b to be in A. This can be problematic, though, since there are (many) sets for which no such point exists. (i.e. $(0,1)$, $\mathbb{R}$). This leads us to the following definition.

Definition 10 *A point b is said to be the* ***least upper bound (supremum)*** *for the set A if b is an upper bound (i.e. $\forall a\in A$, $a\leq b$) and it is the smallest upper bound (i.e. $\forall b'$ satisfying $a\leq b'$ for all $a\in A$, we have that $b\leq b'$).*

We write $b=\sup_{a\in A}a\equiv\sup A$.

This definition is much more robust since every set in $\mathbb{R}$ has a supremum. In addition, the supremum may take on the values $\pm\infty$. (i.e. $\sup(0,1)=1$ and $\sup\mathbb{R}=+\infty$) By convention, we say that if a set contains no elements, its supremum is $-\infty$. That is, $\sup\emptyset=-\infty$.

2.2 Supremum of a Function

Definition 11 *Let $f:\mathbb{R}^n\to\mathbb{R}$. The* ***supremum*** *of f on $Dom(f)$ is defined as $\sup_{x\in Dom(f)}f(x)=\sup\{f(x):x\in Dom(f)\}$.*

Note that the definition of the supremum of a function ends up being exactly the same as that of the definition of the supremum of a set.

3 Homogeneity, Additivity

3.1 Homogeneity

Definition 12 *A function $f:\mathbb{R}^n\to\mathbb{R}$ is said to be* ***positively homogeneous*** *(of degree one) if for all $\lambda>0$, $f(\lambda x)=\lambda f(x)$.*

Definition 13 *A function $f:\mathbb{R}^n\to\mathbb{R}$ is said to be* ***negatively homogeneous*** *(of degree one) if for all $\lambda<0$, $f(\lambda x)=\lambda f(x)$.*

Definition 14 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be* ***homogeneous of degree one*** *if for all λ, $f(\lambda x) = \lambda f(x)$.*

Example 15 (Cobb-Douglas) *Consider the function $f(x,y) = x^\alpha y^{1-\alpha}$. Then*

$$\begin{aligned} f(\lambda x, \lambda x) &= (\lambda x)^\alpha (\lambda y)^{1-\alpha} \\ &= \lambda^\alpha x^\alpha \lambda^{1-\alpha} y^{1-\alpha} \\ &= \lambda x^\alpha y^{1-\alpha} \\ &= \lambda f(x,y) \end{aligned}$$

Therefore, f is homogeneous of degree one.

More generally, we have the following definition.

Definition 16 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be* ***homogeneous of degree*** *k if for all λ, $f(\lambda x) = \lambda^k f(x)$.*

Example 17 (CES) *Consider the function $f(x,y) = \left(x^{1/\rho} + y^{1/\rho}\right)^{k\rho}$. Then*

$$\begin{aligned} f(\lambda x, \lambda y) &= \left((\lambda x)^{1/\rho} + (\lambda y)^{1/\rho}\right)^{k\rho} \\ &= \left(\lambda^{1/\rho} x^{1/\rho} + \lambda^{1/\rho} y^{1/\rho}\right)^{k\rho} \\ &= \left(\lambda^{1/\rho}\left(x^{1/\rho} + y^{1/\rho}\right)\right)^{k\rho} \\ &= \left(\lambda^{1/\rho}\right)^{k\rho}\left(x^{1/\rho} + y^{1/\rho}\right)^{k\rho} \\ &= \lambda^k \left(x^{1/\rho} + y^{1/\rho}\right)^{k\rho} \\ &= \lambda^k f(x,y) \end{aligned}$$

3.2 Additivity

There are three definitions related to additivity.

Definition 18 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be* ***superadditive*** *if for all a, a', we have that $f(a + a') \geq f(a) + f(a')$.*

Example 19 *Let $f(x,y) = xy$, where $x, y \geq 0$. Then*

$$\begin{aligned} f(x_1 + x_2, y_1 + y_2) &= (x_1 + x_2)(y_1 + y_2) \\ &= x_1 y_1 + x_2 y_2 + \underbrace{x_1 y_2}_{\geq 0} + \underbrace{x_2 y_1}_{\geq 0} \\ &\geq f(x_1, y_1) + f(x_2, y_2) \end{aligned}$$

Therefore, f is superadditive.

Definition 20 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be* ***subadditive*** *if for all a, a', we have that $f(a + a') \leq f(a) + f(a')$.*

Example 21 *Let $f(x) = ||x||$, where $||x||$ is a norm. Then, we have that*

$$\begin{aligned} f(x_1 + x_2) &= ||x_1 + x_2|| \leq ||x_1|| + ||x_2|| \\ &= f(x_1) + f(x_2) \end{aligned}$$

That is, f is subadditive.

Definition 22 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be **additive** if for all a, a', we have that $f(a + a') = f(a) + f(a')$.*

Example 23 *Any linear functional is additive. Let $f(x) = Ax$, where A is a matrix. Then we have*

$$\begin{aligned} f(x_1 + x_2) &= A(x_1 + x_2) \\ &= Ax_1 + Ax_2 \\ &= f(x_1) + f(x_2) \end{aligned}$$

Therefore, f is additive.

3.3 Combining the two notions

Proposition 24 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be superadditive and positively homogeneous. Then f is concave.*

Proof. Let $\lambda \in [0, 1]$, $x, x' \in \mathbb{R}^n$ be arbitrary. Then

$$\begin{aligned} f((1-\lambda)x + \lambda x') &\geq f((1-\lambda)x) + f(\lambda x') \\ &= (1-\lambda) f(x) + \lambda f(x') \end{aligned}$$

That is, f is a concave function. ■

Proposition 25 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be subadditive and positively homogeneous. Then f is convex.*

Proof. Let $\lambda \in [0, 1]$, $x, x' \in \mathbb{R}^n$ be arbitrary. Then

$$\begin{aligned} f((1-\lambda)x + \lambda x') &\leq f((1-\lambda)x) + f(\lambda x') \\ &= (1-\lambda) f(x) + \lambda f(x') \end{aligned}$$

That is, f is a convex function. ■