Scaling up/Scaling down

From assignment model to GELP

$$\hat{v}(\bar{z}) = \max_{(\sup)} \left\{ \sum_{k=1}^{l+1} \lambda_k v(z_k) : \lambda_k \geq 0, \sum_{k=1}^{l+1} \lambda_k = 1, \sum_{k=1}^{l+1} \lambda_k z_k = \bar{z} \right\}$$

If $v(z) = z^2$, and we let $\bar{z} = 1$, $\hat{v}(\bar{z}) = +\infty$. To see this, put $\frac{k}{k+1}$ at the allocation $\frac{1}{k}$ and $\frac{1}{k+1}$ at the allocation k

$$\Rightarrow \frac{k}{k+1} \cdot \frac{1}{k} + \frac{1}{k+1} k = 1 \Rightarrow \lambda\left(\tfrac{1}{k}\right) + (1-\lambda) k = 1 \quad \text{(feasible)}$$

But the "expected utility" is

$$\frac{k}{k+1}\left(\frac{1}{k}\right)^2 + \frac{1}{k+1} k^2 \longrightarrow \infty \quad \text{as } k \to \infty$$

The convexifying effect does not work here.

$$h^*(k+1, (k+1)\bar{z}) = \max \Big\{ \sum_z v(z) \underbrace{x_z}_{\text{level of operation of activity } z} : \sum_z x_z = k+1, \sum_z z x_z = (k+1)\bar{z}, x_z \in \{0,1,2,\ldots\} \Big\}$$

Let $\bar{z} = 1$. Clearly, $h^*(k+1, k+1) \xrightarrow{k \to \infty} \infty$. What about $\frac{h^*(k, k\bar{z})}{k}$, the per-capita gains from trade?

$$\frac{h^*(k, k\bar{z})}{k} = \max \left\{ \sum_z v(z) \frac{x_z}{k} : \sum_z \frac{x_z}{k} = 1, \sum_z z \frac{x_z}{k} = \bar{z}, \frac{x_z}{k} \in \left\{0, \tfrac{1}{k}, \tfrac{2}{k}, \ldots\right\} \right\}$$

choosing fractions of population, not # of people.

Proposition: $\lim_{k \to \infty} \frac{h^*(k, k\bar{z})}{k} = \hat{v}(\bar{z})$.

As $k \to \infty$, then $\frac{x_z}{k}$ becomes infinitessimal and we can obtain any $\frac{x_z}{k} \geq 0$.

"The scale of each individual" becomes $\frac{1}{k}$. In the limit, each individual becomes an infinitessimal.

So that the infinitely replicated economy makes sense, scale down the individual.

Define $\lambda_z = \frac{x_z}{k}$.

$$\frac{h^*(k, k\bar{z})}{k} = \max\left\{\sum_z \lambda_z V(z) : \sum_z \lambda_z = 1, \sum_z \lambda_z z = \bar{z}, \lambda_z \in \{0, \tfrac{1}{k}, \tfrac{2}{k}, \ldots\}\right\}$$

As $k \to \infty$, we think of this as a continuum economy with mass one of individuals. Here, we can let $\lambda_z \in [0,1]$ be <u>any</u> real number between 0 and 1.

Assignment model

$$\max \sum_s \sum_b V(b,s)\, x(b,s)$$

$$\text{s.t.} \quad \sum_s x(b,s) \le 1 \quad \forall b$$

$$\sum_b x(b,s) \le 1 \quad \forall s$$

$$x(b,s) \ge 0 \quad \forall b,s$$

Since we are allowing $x(b,s) \ge 0$, we are implicitly assuming a continuum economy. $x(b,s) = \frac{1}{2}$ has two interpretations

- b and s are allocating $\frac{1}{2}$ of their "participation" time to each other. This is not how we interpret this here.
- There are a continuum of each type of individuals and $\frac{1}{2}$ of b's are matched with s's.

In this problem, $x(b,s) \in \{0,1\}$ is not a binding constraint, as a result, we can pretend that we instead have the constraint $x(b,s) \geq 0$.

Suppose $B = \{b\}$ and $S = \{s\}$. (Trivial example)

$$\begin{cases} \max \; V(b,s)\, x(b,s) \\ \quad x(b,s) \leq 1 \\ \quad x(b,s) \leq 1 \\ \quad x(b,s) \geq 0 \end{cases}$$

$$\Leftrightarrow \max \{ c \cdot x : A \underset{x(b,s)}{x} \leq b, \; x \geq 0 \}$$

where $Ax = \begin{array}{c} b \\ s \end{array} \begin{bmatrix} 1 \\ 1 \end{bmatrix} x(b,s) \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

and $c = V(b,s)$, $x = x(b,s)$

There are resource constraints: $x(b,s) \leq 1$. How do we impute prices for those constraints? What is the dual?

$$\min \{ yb : yA \geq c \quad y \geq 0 \}$$

where $y = [y_b \; y_s]$ gives us

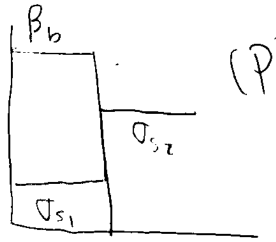
$$\min \{ y_b + y_s : y_b + y_s \geq V(b,s), \; y_b, y_s \geq 0 \}$$

By complementary slackness, if $x(b,s) > 0$, $y_b + y_s = V(b,s)$.
The solution here is any $[y_b \; y_s]$ satisfying $y_b + y_s = V(b,s)$

Optimal solution to primal is $x(b,s)=1$ (assuming that $V(b,s)>0$)

Then $\underbrace{V(b,s)}_{\text{optimal sol'n to primal}} = \underbrace{y_b + y_s}_{\text{optimal sol'n to dual}}$

Let us now consider $B=\{b\}$, $S=\{s_1, s_2\}$.



$$
\begin{aligned}
\text{(P) } \max \quad & V(b,s_1)x(b,s_1) + V(b,s_2)x(b,s_2) \\
\text{s.t.} \quad & x(b,s_1) + x(b,s_2) \le 1 \\
& x(b,s_1) \le 1 \\
& x(b,s_2) \le 1 \\
& x(b,s_1),\ x(b,s_2) \ge 0
\end{aligned}
$$

$$
\begin{aligned}
\text{(D) } \min \quad & y_b + y_{s_1} + y_{s_2} \\
\text{s.t.} \quad & y_b + y_{s_1} \ge V(b,s_1) \\
& y_b + y_{s_2} \ge V(b,s_2) \\
& y_b,\ y_{s_1},\ y_{s_2} \ge 0.
\end{aligned}
$$

The optimal solution to (P) is: $x(b,s_1)=1$, $x(b,s_2)=0$

$\Rightarrow$ the optimal solution to (D) is $y_b + y_{s_1} = V(b,s_2)$

In matrix form,

(P) $\max \{\sum_b \sum_s V(b,s)x(b,s) : Ax \le b,\ x \ge 0\}$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\quad x = \begin{bmatrix} x(b,s_1) \\ x(b,s_2) \end{bmatrix}$$

(D) $\min \{ \sum_b y_b + \sum_s y_s : yA \ge c,\ y \ge 0 \}$

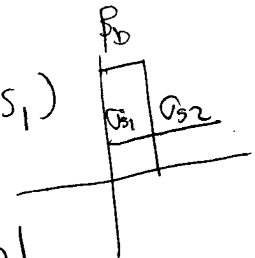
$$y = [y_b \ \ y_{s_1} \ \ y_{s_2}],\quad c = [V(b,s_1) \ \ V(b,s_2)]$$

In this dual, we must have that $y_b + y_s \geq V(b, s_2)$.

$\Rightarrow$ Must give y_b at least as much as he could get for trading with s_2 $\Rightarrow$ $y_b \geq V(b, s_2)$.

What if $V(b, s_2) = V(b, s_1)$? Then the feasible solutions to the dual are:

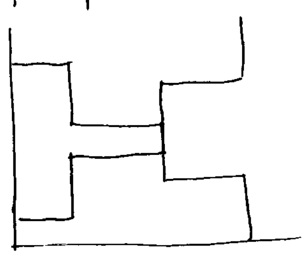
$$y_b + y_{s_1} = V(b, s_1)$$
$$y_b \geq V(b, s_2) = V(b, s_1)$$



$\Rightarrow$ $y_b = V(b, s_1)$, $y_{s_1} = y_{s_2} = 0$.

- Unique solution in which each individual fully appropriates.

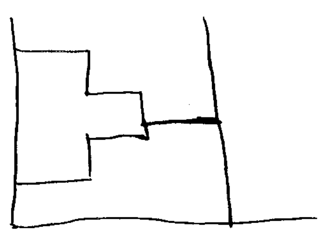
If we have:



Then we have a continuum of solutions to the dual.

$\Rightarrow$ $MP_b \geq y_b$, $MP_s \geq y_s$ (cf $MP_i \geq v_i^*(\varphi)$ in PTE)

If we have:



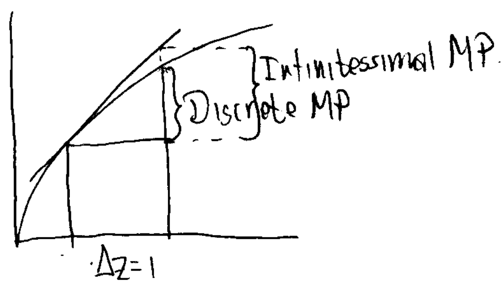
Then there is a unique solution to the dual

$\Rightarrow$ $MP_b = y_b$, $MP_s = y_s$.

$$\max \sum_s \sum_b V(b,s)\, x(b,s)$$
$$\sum_b x(b,s) \leq 1 \quad \forall s$$
$$\sum_s x(b,s) \leq 1 \quad \forall b$$
$$x(b,s) \geq 0 \quad \forall b,s$$

What if we interpret $x(b,s)$ as the mass of individuals of types b and s who are matched.

In the assignment model, the marginal product of an infinitessimal individual is the same as the marginal product of a discrete individual. For a general concave function, discrete MP $\leq$ infinitessimal MP:



For the dual solution, we will have $MP_b^- \geq y_b \geq MP_b^+$

Recall that if $p \in \partial v(z)$, then $-Dv(z;-y) \geq py \geq Dv(z;y)$