- Convexifying Effect of Large Numbers
- Assignment Model
- Does large numbers => PTE? PCE?

Midterm includes today's material.

$v: \mathbb{R}^\ell \to \mathbb{R}$

Define $\hat{v}(\bar{z}) = \max \left\{ \sum_{k=1}^{\ell+1} \lambda_k v(z_k) : \sum_{k=1}^{\ell+1} \lambda_k = 1, \lambda_k \geq 0, \sum_{k=1}^{\ell+1} \lambda_k z_k = \bar{z} \right\}$

- This is the "concavified" function of v. It is the smallest concave function which lies above v.

If v is concave, then $v(z) = \hat{v}(z)\ \forall z$.

If v is not concave, then $v(z) \leq \hat{v}(z)\ \forall z$ and for some z', it may be that $v(z') < \hat{v}(z')$

$h(1,\bar{z}) = \max \left\{ \sum_z v(z) x_z : x_z \geq 0, \sum_z x_z = 1, \sum_z x_z z = \bar{z} \right\}$

Claim: $h(1,\bar{z}) = \hat{v}(\bar{z})$

If v is concave, then $h(2, 2\bar{z}) \overset{x_z = 2 \text{ if } z=\bar{z}}{=} 2h(1,\bar{z})$ ($x_z = 1$ if $z = \bar{z}$)

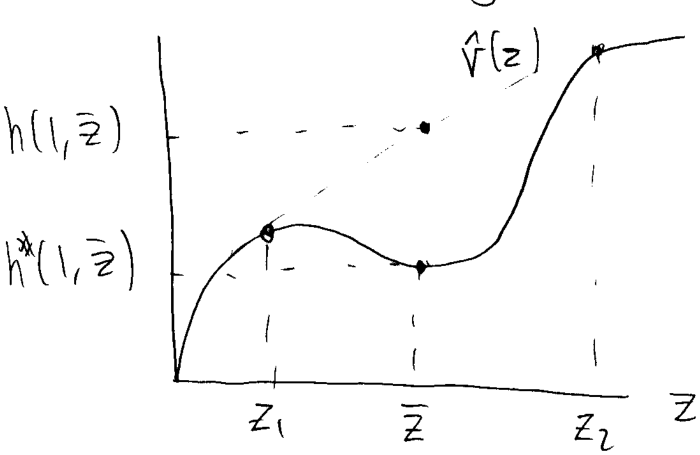
similarly, $h(k, k\bar{z}) = k\,h(1,\bar{z})$

Suppose instead, we make the restriction $x_z \in \{0,1,2,\ldots\}$

$h^*(1,\bar{z}) = \max \left\{ \sum_z v(z) x_z : x_z \in \{0,1,2,\ldots\}, \sum_z x_z = 1, \sum_z x_z z = \bar{z} \right\}$

If v is concave, then $h^*(1,\bar{z}) = h(1,\bar{z})$.

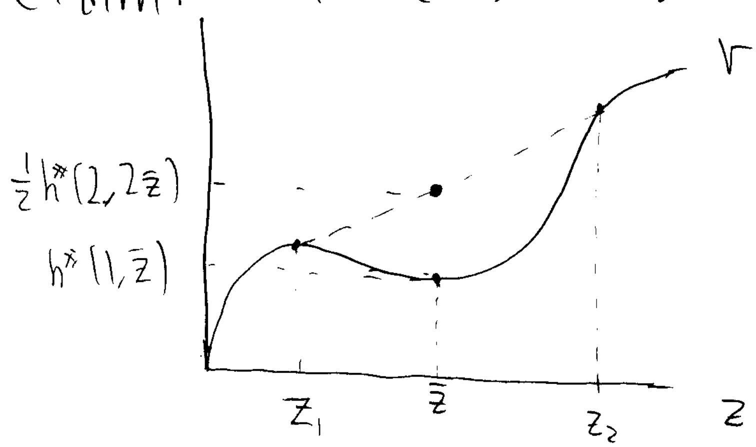
Suppose v is not concave: Then if we restrict ourselves to integer solns: i.e. $x_z \in \{0, 1, \ldots\}$ $\forall z$, we may have that $h^*(1, \bar{z}) < h(1, \bar{z})$.



where $\lambda z_1 + (1-\lambda) z_2 = \bar{z}$ for <u>some</u> $\lambda \in [0,1]$ (not necessarily an integer)

What about $h^*(2, 2\bar{z})$?

Claim: $h^*(2, 2\bar{z}) \geq 2 h^*(1, \bar{z})$.



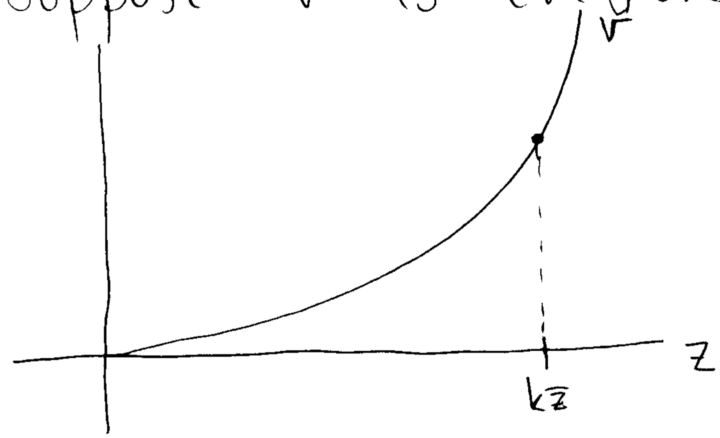
where $1 \cdot z_1 + 1 \cdot z_2 = \bar{z}$

$\Rightarrow \frac{1}{2} h^*(2, 2\bar{z}) > h^*(1, \bar{z}) \Rightarrow h^*(2, 2\bar{z}) > 2h^*(1, \bar{z})$

<u>Claim:</u> for any $k \geq 1$, $h^*(k, k\bar{z}) \geq k\, h^*(1, \bar{z})$

We know that $h^*(200, 200\bar{z}) \geq 2 h^*(100, 100\bar{z})$ and $h^*(200, 200\bar{z}) - 2h^*(100, 100\bar{z})$ is small. also, $h^*(1000, 1000\bar{z}) - 2h^*(500, 500\bar{z})$ is smaller.

Suppose v is everywhere convex:



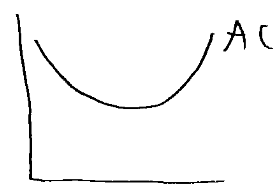
Here, we want to put all the resources into one factory.

When v is everywhere strictly convex,

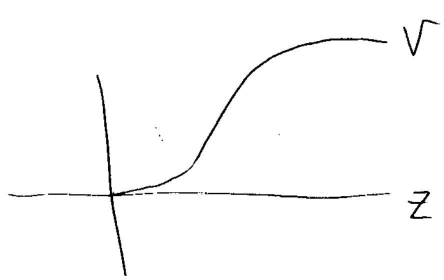
$h^*(k, k\bar{z}) > k\, h^*(1, \bar{z})$

i.e. if v has unbounded nonconvexities (if $\hat{v}$ does not exist), then $\frac{1}{k} h^*(k, k\bar{z}) \to \infty$ as $k \to \infty$

Recall: In intermediate micro:

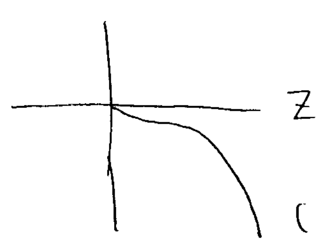


$\Rightarrow$



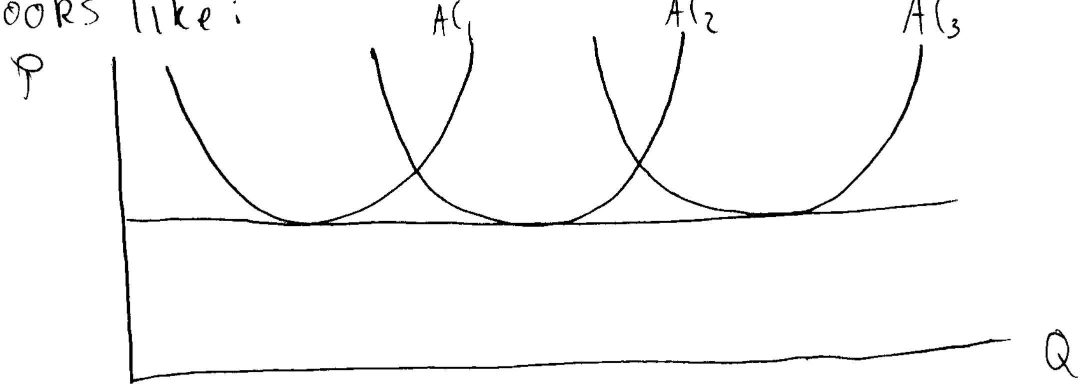
This is the v corresponding to the above AC diagram
- Increasing returns to scale initially
- Decreasing returns to scale eventually

$\Rightarrow$



Increasing marginal cost.

Recall: In intermediate micro, the industry supply curve looks like:



° Horizontal industry supply curve if k is large.

Claim: If v is concave, then $h^*(k, k\bar{z}) = h(k, k\bar{z})$

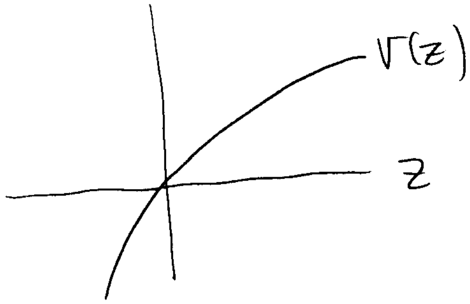
Defn: If $h^*(k,k\bar{z}) = h(k, k\bar{z})$ (ie the optimal solution for h is in the integers.), we say that the economy is integrally optimal

Defn: If $h(k,k\bar{z}) = k h(1,\bar{z})$ (ie h is positively homogeneous), we say that the economy is replica invariant.

Claim: $h^*(k,k\bar{z}) = h(k,k\bar{z}) \Leftrightarrow h(k,k\bar{z}) = k h(1,\bar{z})$
ie integral optimality is equivalent to replica invariance.

When we have unbounded nonconvexities, we do not have the convexifying effect of large numbers.

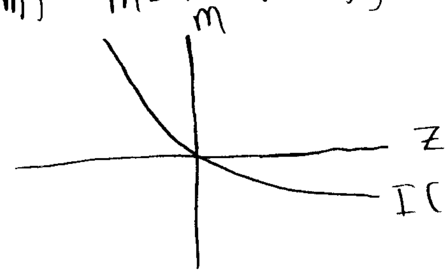
What are we "convexifying"?
° The "better-than" sets.



What does an indifference curve look like?

$IC = \{ (z,m) : v(z) + m = k \}$
$= \{ (z,m) : m = k - v(z) \}$

$\Rightarrow$



Claim: $U(z,m) = v(z) + m$ is quasi concave (convex upper contour sets) if and only if v is concave.

Convexity is "necessary" for the existence of PTEs only when you have small numbers. What we really need is the existence of a separating hyperplane.

Assignment Model vs commodity representation of assignment model.

Defn: An assignment model is a triple $(B, S, V(b,s))$ of buyers $B = \{1, \dots, n\}$, sellers $S = \{1, \dots, m\}$, and values of matches $V(b,s) : B \times S \to \mathbb{R}_+$.

We do not specify the source of the values of matches.

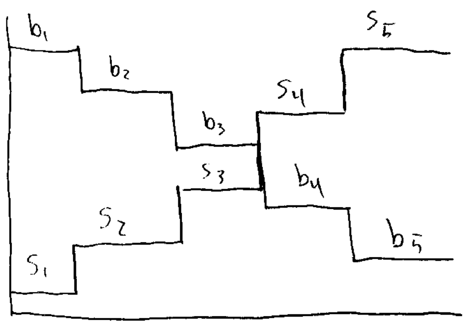
If $n > m$, (ie there are more buyers than sellers):

$$\begin{array}{cc} & s \\ b & \left[\begin{array}{ccc|c} & \vdots & & 0 \\ \cdots & V(b,s) & \cdots & \vdots \\ & \vdots & & 0 \end{array}\right] \end{array}$$

we have a non-square matrix. To remedy this, add $n-m$ dummy sellers, with $V(b,s)=0$ for all those sellers.

One motivation for $V(b,s)$ is that buyers and sellers have reservation values:

<u>Define</u>: $V(b,s) = \max\{b-s, 0\}$



commodity representation

Can we represent <u>all</u> assignment models with a commodity representation?

- The answer is no. There exist matrices $V(b,s)$ that cannot be represented.

Equilibrium in the assignment model:

An assignment $\{x(b,s)\}_{b,s}$ is <u>feasible</u> if

- $\sum_b x(b,s) \leq 1 \quad \forall s$ (each seller can have at most one match)
- $\sum_s x(b,s) \leq 1 \quad \forall b$ (each buyer can have at most one match
- and $x(b,s) \in \{0,1\} \; \forall s, \forall b$ (integer constraints)

where $x(b,s) = \begin{cases} 1 & \text{if } b \text{ and } s \text{ are matched} \\ 0 & \text{if } b \text{ and } s \text{ are not matched} \end{cases}$

If $n=m$, then we have equality.

- $\sum_b x(b,s)=1 \quad \forall s$
- $\sum_s x(b,s)=1 \quad \forall b$
- $x(b,s)\in\{0,1\} \quad \forall b, \forall s$

If $x(b,s)=1$, then b and s get to share $V(b,s)$. This is a multilateral bargaining problem (there are outside options).

Let $y_b \equiv$ amount buyer receives
$y_s \equiv$ amount seller receives

Suppose $y_b + y_s = V(b,s)$
$y_{b'} + y_{s'} = V(b',s')$

and $y_b + y_{s'} < V(b,s')$

$\Rightarrow$ It is in b and s' best interest to form a match. That is, $\exists\ \bar{y}_b + \bar{y}_{s'} = V(b,s')$ with $\bar{y}_b > y_b$ and $\bar{y}_{s'} > y_{s'}$.

If this is possible, we are <u>not</u> in equilibrium.

<u>Defn</u>: a vector $[\{y_b\}_b, \{y_s\}_s, \{x(b,s)\}_{b,s}]$ is an <u>equilibrium</u> if
- $\{x(b,s)\}_{b,s}$ is feasible
- $y_b + y_s \geq V(b,s) \quad \forall b,s$ (no "better" matches)
- $x(b,s)=1 \Rightarrow y_b + y_s = V(b,s)$. (feasible split)

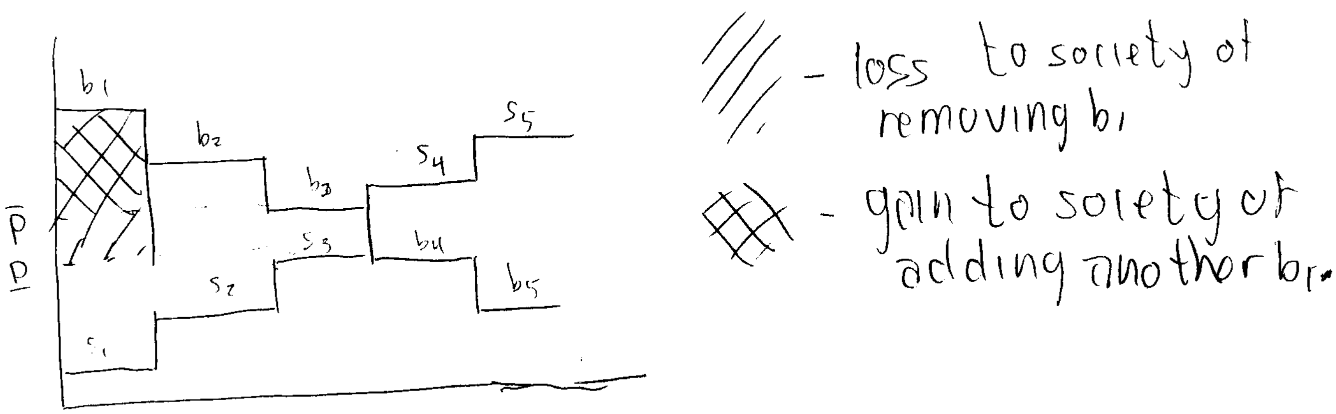
The primal is:

$$(P) \max \sum_{b,s} V(b,s)\, x(b,s)$$

$$\text{s.t.} \quad \sum_{b} x(b,s) \le 1 \quad \forall s$$

$$\sum_{s} x(b,s) \le 1 \quad \forall b$$

$$x(b,s) \ge 0 \quad \forall b,s.$$



Here, $\sum_{i \in BUS} MP_i > V_I$

Anything that will give us a unique equilibrium price will lead to $\sum_{i \in BUS} MP_i = V_I$

The more indeterminacy of prices there is in the assignment model, the larger the gap between $\sum_{i \in BUS} MP_i$ and V_I.