

## Econ 201C: General Equilibrium and Welfare Economics

### Flattening Effect Examples

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## 1 Simple Example of the Flattening Effect

The goal of this question is to provide an analytical example of the flattening effect of large numbers.

(a) Define  $v_2(z) = [v \boxplus v](z)$ . Compute  $v_2(z)$  for preferences given by  $v(z) = 2z - \frac{1}{2}z^2$ .

**Solution:** I will begin by deriving a more general result that can be easily applied to all strictly concave, differentiable functions.

$$\begin{aligned}v_2(z) &= [v \boxplus v](z) = \max_{z_1, z_2} \{v(z_1) + v(z_2) : z_1 + z_2 = z\} \\ &= \max_{z_1} \{v(z_1) + v(z - z_1)\}\end{aligned}$$

Taking first order conditions,

$$(z_1) : v'(z_1) + v'(z - z_1)(-1) = 0$$

$$v'(z_1) = v'(z - z_1)$$

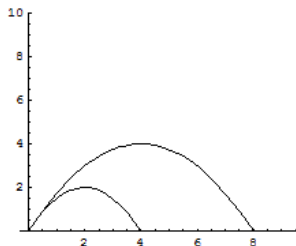
If  $v$  is strictly concave, then this implies that

$$\begin{aligned}z_1 &= z - z_1 \\ z_1 &= \frac{1}{2}z\end{aligned}$$

Therefore,

$$\begin{aligned}v_2(z) &= v\left(\frac{1}{2}z\right) + v\left(\frac{1}{2}z\right) = 2v\left(\frac{1}{2}z\right) \\ &= 2\left(2\frac{z}{2} - \frac{1}{2}\left(\frac{z}{2}\right)^2\right) \\ &= 2z - \frac{z^2}{4}\end{aligned}$$

Graphing both of these functions,



(b) For a general concave function, show that  $v_k(z) = kv\left(\frac{z}{k}\right)$ .

**Solution:**

$$\begin{aligned}
 v_k(z) &= [v \boxplus \dots \boxplus v](z) \\
 &= \max_{z_1, \dots, z_k} \{v(z_1) + \dots + v(z_k) : z_1 + \dots + z_k = z\} \\
 &= \max_{z_1, \dots, z_{k-1}} \{v(z_1) + \dots + v(z_{k-1}) + v(z - z_1 - \dots - z_{k-1})\}
 \end{aligned}$$

Taking first order conditions

$$\begin{aligned}
 (z_1) \quad &: \quad v'(z_1) = v'(z - z_1 - \dots - z_{k-1}) \\
 &\quad \vdots \\
 (z_{k-1}) \quad &: \quad v'(z_{k-1}) = v'(z - z_1 - \dots - z_{k-1})
 \end{aligned}$$

By strict concavity, this implies that

$$\begin{aligned}
 z_1 &= z - z_1 - \dots - z_{k-1} \\
 &\quad \vdots \\
 z_{k-1} &= z - z_1 - \dots - z_{k-1}
 \end{aligned}$$

Or

$$z_1 = \dots = z_{k-1}$$

Since  $z_k = z - z_1 - \dots - z_{k-1}$ , we have that

$$z_1 = \dots = z_k = \frac{z}{k}$$

Therefore,

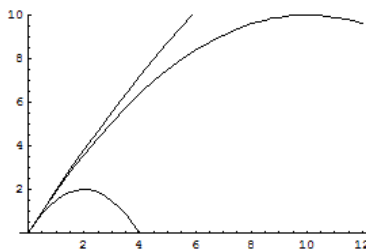
$$\begin{aligned}
 v_k(z) &= v\left(\frac{z}{k}\right) + \dots + v\left(\frac{z}{k}\right) \\
 &= kv\left(\frac{z}{k}\right)
 \end{aligned}$$

(c) For these preferences, compute  $v_k(z)$ .

**Solution:**

$$\begin{aligned}
 v_k(z) &= kv\left(\frac{z}{k}\right) \\
 &= k\left(2\frac{z}{k} - \frac{1}{2}\left(\frac{z}{k}\right)^2\right) \\
 &= 2z - \frac{1}{2k}z^2
 \end{aligned}$$

Plotting these functions for  $k = 1, 5$ , and  $10$ , the flattening effect of large numbers becomes apparent.



(d) Confirm that for a general strictly concave  $v$  with  $v(0) = 0$ ,

$$v_{\infty}(z) \equiv \lim_{k \rightarrow \infty} v_k(z) = Dv(0; z).$$

**Solution:** Recall from part (b)

$$v_k(z) = kv\left(\frac{z}{k}\right) = \frac{v\left(0 + \frac{1}{k}z\right) - v(0)}{\frac{1}{k}}$$

Taking limits as  $k \rightarrow \infty$ ,

$$\begin{aligned} \lim_{k \rightarrow \infty} v_k(z) &= \lim_{k \rightarrow \infty} \frac{v\left(0 + \frac{1}{k}z\right) - v(0)}{\frac{1}{k}} \\ &= \lim_{t \downarrow 0} \frac{v(0 + tz) - v(0)}{t} \\ &\equiv Dv(0; z) \end{aligned}$$

(e) For the preferences given above, compute  $v_{\infty}(z)$ .

**Solution:** Here, we have

$$\begin{aligned} v_{\infty}(z) &= Dv(0; z) = \lim_{t \downarrow 0} \frac{v(0 + tz) - v(0)}{t} \\ &= \lim_{t \downarrow 0} \frac{2tz - \frac{1}{2}(tz)^2}{t} \\ &= \lim_{t \downarrow 0} 2z - \frac{1}{2}tz^2 \\ &= 2z \end{aligned}$$

## 2 Uncentered Flattening Effect

In the previous question, we saw an example of the flattening effect. Implicitly, in this example, we flattened  $v$  "around the origin." Suppose instead we want to flatten  $v$  around a different point in its domain. Let  $v$  be strictly concave.

(a) Define  $\Delta v(z; \bar{z}) = v(\bar{z} + z) - v(\bar{z})$ . Compute  $\Delta v_2(z; \bar{z}) \equiv [\Delta v(\cdot; \bar{z}) \boxplus \Delta v(\cdot; \bar{z})](z)$

**Solution:**

$$\begin{aligned} \Delta v_2(z; \bar{z}) &= [\Delta v(\cdot; \bar{z}) \boxplus \Delta v(\cdot; \bar{z})](z) \\ &= \max_{z_1, z_2} \{ \Delta v(z_1; \bar{z}) + \Delta v(z_2; \bar{z}) : z_1 + z_2 = z \} \\ &= \max_{z_1, z_2} \{ v(\bar{z} + z_1) - v(\bar{z}) + v(\bar{z} + z_2) - v(\bar{z}) : z_1 + z_2 = z \} \\ &= \max_{z_1} \{ v(\bar{z} + z_1) + v(\bar{z} + z - z_1) \} - 2v(\bar{z}) \end{aligned}$$

Taking first order conditions,

$$(z_1) : v'(\bar{z} + z_1) = v'(\bar{z} + z - z_1)$$

By strict concavity of  $v$ , we have

$$\begin{aligned} \bar{z} + z_1 &= \bar{z} + z - z_1 \\ z_1 &= \frac{1}{2}z \end{aligned}$$

This gives us

$$\begin{aligned} \Delta v_2(z; \bar{z}) &= v\left(\bar{z} + \frac{1}{2}z\right) + v\left(\bar{z} + \frac{1}{2}z\right) - 2v(\bar{z}) \\ &= 2\left[v\left(\bar{z} + \frac{1}{2}z\right) - v(\bar{z})\right] \end{aligned}$$

(b) Generalize this result to show that  $\Delta v_k(z; \bar{z}) = k[v(\bar{z} + \frac{z}{k}) - v(\bar{z})]$ .

**Solution:**

$$\begin{aligned} \Delta v_k(z; \bar{z}) &= [\Delta v(\cdot; \bar{z}) \boxplus \dots \boxplus \Delta v(\cdot; \bar{z})](z) \\ &= \max_{z_1, \dots, z_k} \{ \Delta v(z_1; \bar{z}) + \dots + \Delta v(z_k; \bar{z}) : z_1 + \dots + z_k = z \} \\ &= \max_{z_1, \dots, z_{k-1}} \{ \Delta v(z_1; \bar{z}) + \dots + \Delta v(z_{k-1}; \bar{z}) + \Delta v(z - z_1 - \dots - z_{k-1}; \bar{z}) \} \\ &= \max_{z_1, \dots, z_{k-1}} \{ v(\bar{z} + z_1) - v(\bar{z}) + \dots + v(\bar{z} + z_{k-1}) - v(\bar{z}) + v(\bar{z} + z - z_1 - \dots - z_{k-1}) - v(\bar{z}) \} \\ &= \max_{z_1, \dots, z_{k-1}} \{ v(\bar{z} + z_1) + \dots + v(\bar{z} + z_{k-1}) + v(\bar{z} + z - z_1 - \dots - z_{k-1}) \} - kv(\bar{z}) \end{aligned}$$

Taking first order conditions,

$$\begin{aligned} (z_1) : v'(\bar{z} + z_1) &= v'(\bar{z} + z - z_1 - \dots - z_{k-1}) \\ &\vdots \\ (z_{k-1}) : v'(\bar{z} + z_{k-1}) &= v'(\bar{z} + z - z_1 - \dots - z_{k-1}) \end{aligned}$$

By strict concavity of  $v$ , we have that

$$\begin{aligned} \bar{z} + z_1 &= \bar{z} + z - z_1 - \dots - z_{k-1} = \bar{z} + z_k \\ &\vdots \\ \bar{z} + z_{k-1} &= \bar{z} + z - z_1 - \dots - z_{k-1} = \bar{z} + z_k \end{aligned}$$

Or

$$z_1 = \cdots = z_k = \frac{1}{k}z$$

Therefore

$$\begin{aligned}\Delta v_k(z; \bar{z}) &= v\left(\bar{z} + \frac{1}{k}z\right) + \cdots + v\left(\bar{z} + \frac{1}{k}z\right) - kv(\bar{z}) \\ &= k\left[v\left(\bar{z} + \frac{1}{k}z\right) - v(\bar{z})\right]\end{aligned}$$

(c) For the function  $v = z^{1/2}$ , compute  $\Delta v_k(z; \bar{z})$ .

**Solution:** Using the result from part (b), we have

$$\begin{aligned}\Delta v_k(z; \bar{z}) &= k\left[v\left(\bar{z} + \frac{1}{k}z\right) - v(\bar{z})\right] \\ &= k\left[\left(\bar{z} + \frac{1}{k}z\right)^{\frac{1}{2}} - (\bar{z})^{\frac{1}{2}}\right]\end{aligned}$$

Letting  $z = 4$ , this becomes

$$\Delta v_k(z; \bar{z}) = k\left[\left(4 + \frac{1}{k}z\right)^{\frac{1}{2}} - 4^{\frac{1}{2}}\right]$$

(d) Show that  $\lim_{k \rightarrow \infty} \Delta v_k(z; \bar{z}) = Dv(z; \bar{z})$ .

**Solution:** From part (b), we have

$$\begin{aligned}\Delta v_k(z; \bar{z}) &= k\left[v\left(\bar{z} + \frac{1}{k}z\right) - v(\bar{z})\right] \\ &= \frac{v\left(\bar{z} + \frac{1}{k}z\right) - v(\bar{z})}{\frac{1}{k}}\end{aligned}$$

Taking limits as  $k \rightarrow \infty$ ,

$$\begin{aligned}\lim_{k \rightarrow \infty} \Delta v_k(z; \bar{z}) &= \lim_{k \rightarrow \infty} \frac{v\left(\bar{z} + \frac{1}{k}z\right) - v(\bar{z})}{\frac{1}{k}} \\ &= \lim_{t \downarrow 0} \frac{v(\bar{z} + tz) - v(\bar{z})}{t} \\ &= Dv(z; \bar{z}).\end{aligned}$$