Econ 201C: General Equilibrium and Welfare Economics

Differentiation and Subdifferentials

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1 Differentiability and Subdifferentials

1.1 The Difference Quotient

Definition 1 *Let $f : \mathbb{R}^n \to \mathbb{R}$. The quantity*

$$\varphi_{x,x'}(t) = \frac{f(x + tx') - f(x)}{t}$$

is referred to as the ***difference quotient of*** *f at x in the direction x'.*

As long as $f(x + tx')$ and $f(x)$ exist, we will always have that $\varphi_{x,x'}(t)$ exists. The difference quotient is related to the slope of the secant line connecting the point $(x, f(x))$ and $(x + tx', f(x + tx'))$.

As mentioned in class, if f is a concave function, the difference quotient of f at x in the direction x' is decreasing in t. Heuristically, in the univariate case, this is a result of the fact that for a concave function, the tangent line always lies above the curve. (And therefore, the slope of the tangent line always exceeds that of the secant lines) As t decreases, the slope of the secant line gets closer to the slope of the tangent line. That is, the slope of the secant line increase. In other words, $\varphi_{x,x'}(t)$ is decreasing in t.

In the limit as $t \to 0$, we approach the world of calculus, to which I will now turn.

1.2 Standard Differentiability and the Derivative

Definition 2 *A function $f : \mathbb{R} \to \mathbb{R}$ is* ***differentiable*** *at the point x if*

$$\underbrace{\lim_{t \downarrow 0} \frac{f(x + t) - f(x)}{t}}_{\text{right hand limit}} = \underbrace{\lim_{t \uparrow 0} \frac{f(x + t) - f(x)}{t}}_{\text{left hand limit}}.$$

We define the ***derivative*** *of f at the point x to be this common limit:*

$$f'(x) = \lim_{t \to 0} \frac{f(x + t) - f(x)}{t}$$

Remark 3 *It is quite possible to define differentiability of a multivariable function $f : \mathbb{R}^n \to \mathbb{R}$, but it would not be terribly instructive to do so. Basically, it involves the idea of continuous partial derivatives within a closed disk around a particular point. One implication of the differentiability of a function is that, indeed, left-hand limits and right-hand limits coincide. Though I will not clarify the concept of differentiability of a multivariable function too much, I will at least define what a partial derivative is.*

Definition 4 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at x. Let $e_j = \left(0, \ldots, 0, \underbrace{1}_{j^{th} \text{ term}}, 0, \ldots, 0\right)$. Then we define the* ***partial derivative*** *of f with respect to x_j at the point x to be*

$$\frac{\partial f(x)}{\partial x_j} \equiv \lim_{t \to 0} \frac{f(x + te_j) - f(x)}{t}.$$

Imposing differentiability on a function is quite a strong assumption, especially in the context of this course. As a result, it is necessary to strengthen our understanding of exactly what it means for a function to be differentiable.

Proposition 5 *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at x. Then for x' infinitessimally close to x, we have that*

$$\begin{aligned} f(x') &= f(x) + \frac{df(x)}{dx}(x - x') \\ &= mx + b \end{aligned}$$

That is, locally, f is linear.

Proof. Suppose f is differentiable at x. Then, if we let $t = (x' - x)$

$$\frac{df(x)}{dx} = \lim_{t \to 0} \frac{f(x+t) - f(x)}{t} = \lim_{(x'-x) \to 0} \frac{f(x') - f(x)}{x' - x}$$

Or

$$\lim_{(x'-x) \to 0} f(x') = \lim_{(x'-x) \to 0} \left[f(x) + \frac{df(x)}{dx}(x' - x) \right]$$

Which states that, for x' infinitessimally close to x, f is linear. ∎

A similar proposition holds for a differentiable function of several variables.

1.3 The Directional Derivative

Even if a function is not differentiable at a particular point, it is still possible to characterize the behavior of the function locally with respect to a particular direction. Consider the following single-variable function:

$$f(x) = \begin{cases} x & -\infty < x \leq 1 \\ 1 & x > 1 \end{cases}$$

I will now show that f is not differentiable at 1.

$$\lim_{t \uparrow 0} \frac{f(1+t) - f(1)}{t} = \lim_{t \uparrow 0} \frac{1 + t - 1}{t} = 1 \text{ (since if } t < 0,\ 1 + t < 1 \text{ and thus } f(1+t) = 1 + t)$$

$$\lim_{t \downarrow 0} \frac{f(1+t) - f(1)}{t} = \lim_{t \downarrow 0} \frac{1 - 1}{t} = 0$$

Here, we have that

$$\lim_{t \uparrow 0} \frac{f(1+t) - f(1)}{t} \neq \lim_{t \downarrow 0} \frac{f(1+t) - f(1)}{t}$$

And therefore, f is not differentiable at 1.

This does not preclude us from saying what happens around the point 1 as we "move to the right" or "move to the left" however. Here, the "limit from the right" is well-defined (and equal to zero), as is the "limit from the left." This concept can be generalized to the notion of the directional derivative.

Definition 6 *Let $f : \mathbb{R}^n \to \mathbb{R}$. We refer to*

$$Df(x; a) = \lim_{t \downarrow 0} \frac{f(x + at) - f(x)}{t}$$

*as the **directional derivative** of f at x in the direction a.*

Specializing this definition to our particular example, we see that

$$\begin{aligned} Df(1;1) &= \lim_{t\downarrow 0}\frac{f(1+t)-f(1)}{t}=0 \\ Df(1;-1) &= \lim_{t\downarrow 0}\frac{f(1-t)-f(1)}{t}=\lim_{t\downarrow 0}\frac{1-t-1}{t}=-1 \end{aligned} \tag{1}$$

It is important to note that in the definition of the directional derivative, we are restricting $t>0$. The fact that in (1) we are approaching from the left is captured by the "direction" we are taking the derivative with respect to. For differentiable functions, there are some nice properties of the directional derivative.

Proposition 7 *Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable at x. Then we have that*

$$Df(x;a)=-Df(x;-a)$$

Proof. Since f is differentiable at x, we have that the "right hand limit" and the "left hand limit" coincide.

$$Df(x;a)=\lim_{t\downarrow 0}\frac{f(x+at)-f(x)}{t}=\lim_{t\uparrow 0}\frac{f(x+at)-f(x)}{t}$$

Define $s=-t$. Then

$$\begin{aligned} \lim_{t\uparrow 0}\frac{f(x+at)-f(x)}{t} &= \lim_{s\downarrow 0}\frac{f(x+a(-s))-f(x)}{(-s)} \\ &= -\lim_{s\downarrow 0}\frac{f(x+s(-a))-f(x)}{s} \\ &= -Df(x;-a) \end{aligned}$$

Which is the desired result. ■

As we saw in the above example, this proposition does not hold if f is not differentiable. This idea is actually quite generalizable. It turns out that for a differentiable function f, the directional derivative is homogeneous of degree one with respect to the direction. That is, for all λ, $Df(x;\lambda a)=\lambda Df(x;a)$. In order to establish this, I must first prove the following result. (The proof is not relevant for this class, but will be included for completeness.)

Proposition 8 *Let $f:\mathbb{R}^n\to\mathbb{R}$ be differentiable at $x=(x_1,\ldots,x_n)$. Let $a=(a_1,\ldots,a_n)$ be a vector. Then*

$$\lim_{t\to 0}\frac{f(x_1+a_1t,\ldots,x_n+a_nt)-f(x_1,\ldots,x_n)}{t}=a_1\frac{\partial f(x)}{\partial x_1}+\cdots+a_n\frac{\partial f(x)}{\partial x_n}$$

Sketch of Proof. Here, I will consider the case where $n=2$, and I will not be too careful about what exactly it means for a multivariable function to be differentiable.

$$\begin{aligned} & \lim_{t\to 0}\frac{f(x_1+a_1t,x_2+a_2t)}{t} \\ = & \lim_{t\to 0}\frac{f(x_1+a_1t,x_2+a_2t)-f(x_1,x_2+a_2t)}{t}+\lim_{t\to 0}\frac{f(x_1,x_2+a_2t)-f(x_1,x_2)}{t} \end{aligned} \tag{2}$$

By the mean value theorem, we have that

$$f(x_1+a_1t,x_2+a_2t)=f(x_1,x_2+a_2t)+\frac{\partial f}{\partial x_1}(\xi,x_2+a_2t)(x_1+a_1t-x_1)$$

Where $\xi\in|x_1,x_1+a_1t|$. Rearranging,

$$f(x_1+a_1t,x_2+a_2t)-f(x_1,x_2+a_2t)=\frac{\partial f}{\partial x_1}(\xi,x_2+a_2t)\,a_1t$$

And therefore,

$$\begin{aligned}\lim_{t\to 0}\frac{f(x_1+a_1t,x_2+a_2t)-f(x_1,x_2+a_2t)}{t} &= a_1\lim_{t\to 0}\frac{\partial f}{\partial x_1}(\xi,x_2+a_2t)\\ &= a_1\frac{\partial f}{\partial x_1}(x_1,x_2) \qquad (3)\end{aligned}$$

Where, since $\xi\in|x_1,x_1+a_1t|$, we have that $\lim_{t\to 0}\xi=x_1$. Also, note that if we define $s=a_2t$, we have that

$$\lim_{t\to 0}\frac{f(x_1,x_2+a_2t)-f(x_1,x_2)}{t}=a_2\lim_{s\to 0}\frac{f(x_1,x_2+s)-f(x_1,x_2)}{s}=a_2\frac{\partial f}{\partial x_2}(x_1,x_2) \qquad (4)$$

Plugging (3) and (4) into (2) gives us:

$$\lim_{t\to 0}\frac{f(x_1+a_1t,x_2+a_2t)}{t}=a_1\frac{\partial f}{\partial x_1}(x_1,x_2)+a_2\frac{\partial f}{\partial x_2}(x_1,x_2)$$

Which is the desired result. ■

This allows us to establish the next result.

Corollary 9 *Let $f:\mathbb{R}^n\to\mathbb{R}$ be differentiable at $x=(x_1,\dots,x_n)$ and let $a=(a_1,\dots,a_n)$ be a vector. Then*

$$\begin{aligned}\lim_{t\to 0}\frac{f(x_1+\lambda a_1t,\dots,x_n+\lambda a_nt)}{t} &= \lambda a_1\frac{\partial f}{\partial x_1}(x)+\cdots+\lambda a_n\frac{\partial f}{\partial x_n}(x)\\ &= \lambda\lim_{t\to 0}\frac{f(x_1+a_1t,\dots,x_n+a_nt)}{t}\end{aligned}$$

We can specialize this corollary for directional derivatives in the following proposition.

Proposition 10 *Let $f:\mathbb{R}^n\to\mathbb{R}$ be differentiable at x. Then*

$$Df(x;\lambda a)=\lambda Df(x;a)$$

For all λ.

Finally, there is a nice homogeneity property of the directional derivative of f even if f is not differentiable at x.

Proposition 11 *Let $f:\mathbb{R}^n\to\mathbb{R}$. Then $Df(x;\lambda a)=\lambda Df(x;a)$ for all $\lambda>0$. That is, $Df(x;\lambda a)$ is positively homogeneous.*

1.4 The Subdifferential

In somewhat of a preview of what is to come, consider the following consumer's optimization problem

$$\max\{v(z)+m : m+p\cdot z=0\}$$

Definition 12 *A consumer with preferences given by $u(z,m)=v(z)+m$ is said to have **quasilinear preferences**.*

Solving the constraint for m, this problem can be written in the following unconstrained optimization form

$$v^*(p)=\max\{v(z)-p\cdot z\}$$

Where we interpret $v(z)$ as being the utility from trading the bundle z and $-p\cdot z$ as the disutility associated with paying for the bundle z.

Definition 13 *The function $v^*(p)=\max\{v(z)-p\cdot z\}$ is referred to as the **conjugate** of v at p. Verbally, $v^*(p)$ is the maximized utility at the prices p and is often referred to as the **indirect utility function**.*

Suppose we know that the consumer optimally chooses the trade z^*. What can we say about the prices that he must have been facing? This is exactly the question that the concept of the subdifferential answers.

Definition 14 *The set $\partial v(z^*) = \{p : v(z^*) - p \cdot z^* \geq v(z) - p \cdot z \ \forall z\}$ is known as the **subdifferential** of v at the bundle z^*. (That is, $\partial v(z^*)$ is the set of all prices for which z^* gives more utility than any other bundle.) The set $\partial v(z^*)$ can also be referred to as the **inverse demand correspondence** for z^*. (i.e. the set of prices supporting z^* being optimal.)*

If v is differentiable (and concave), it will necessarily be the case that for any z^*, there is a unique price vector supporting z^* as an optimum as the next proposition shows.

Proposition 15 *Suppose $v : \mathbb{R}^\ell \to \mathbb{R}$ is differentiable and concave. Then $\partial v(z^*)$ is a singleton for all z^*. Further, $\partial v(z^*) = \{\nabla v(z^*)\}$.*

Proof. Let $v : \mathbb{R}^\ell \to \mathbb{R}$ be differentiable. Then the first order conditions for utility maximization are

$$(z) : \nabla v(z^*) - p = 0$$

Which gives us $p = \nabla v(z^*)$. But what does this equality mean? Facing prices p, the consumer optimally chose the bundle z^*. That is, $p \in \partial v(z^*)$. The first order conditions ensure that there are no $p' \neq p$, $p' \in \partial v(z^*)$. In other words, we have that $\partial v(z^*) = \{p\} = \{\nabla v(z^*)\}$. ■

There is a relationship between the conjugate of a function and the subdifferential, as illustrated in the next proposition.

Proposition 16 *$p \in \partial v(z^*)$ if and only if $v^*(p) = v(z^*) - p \cdot z^*$.*

Proof. $p^* \in \partial v(z^*) = \{p : v(z^*) - p \cdot z^* \geq f(z) - p \cdot z \ \forall z\}$. This occurs if and only if we have that $v(z) - p^* \cdot z = \max\{f(z) - p \cdot z\} \equiv f^*(p^*)$. ■

Finally, it is possible to relate the idea of the subdifferential to that of the directional derivative, which I will do in the following proposition.

Proposition 17 *Let $p \in \partial v(z^*)$. Then $p \cdot z \geq Dv(z^*; z)$ for all z.*

Proof. Let z be arbitrary. Suppose $p \in \partial v(z^*)$. Then we have that for all z',

$$v(z^*) - p \cdot z^* \geq v(z') - p \cdot z'$$

In particular, let $z' = z^* + tz$. Then,

$$v(z^*) - p \cdot z^* \geq v(z^* + tz) - p \cdot (z^* + tz)$$

Or

$$\begin{aligned} p \cdot (z^* + tz) - p \cdot (z^*) &\geq v(z^* + tz) - v(z^*) \\ tp \cdot z &\geq v(z^* + tz) - v(z^*) \\ p \cdot z &\geq \frac{v(z^* + tz) - v(z^*)}{t} \end{aligned}$$

Taking limits, we have that

$$p \cdot z \geq Dv(z^*; z)$$

Since z was arbitrary, this inequality holds $\forall z$. ■

This inequality will be very important in this course. When we conceive of the notion of the directional derivative with respect to an individual in the economy, we will see that this inequality implies that, in general, an individual will be paid no more than that which he contributes to society (and in most cases, this inequality is strict). This gap between contribution and reward, it turns out, is the source of all market failures.