April 4th, 2005 – Math 171 Lecture 1

Def: A stochastic process is a collection of random variables indexed by time.
Eg: Discrete time X_n, $n = 0,1,2,...$ Chapter 1
Eg: Continuous time X_t, $t \geq 0$ Chapter 4

Example: Gambler's Ruin

$0 < p < 1$

X_n = assets of the gambler at time n

$N > 0$ given. Take some $0 \leq i \leq N$. $X_0 = i$

The gamble: At each time t,
Win $1 with probability p
Lose $1 with probability $1 - p$

Continue until $X_n = 0$ $X_n = N$. Then you stop.

Mathematically, this says:

$$P(X_{n+1} = j \mid X_1 = i_1, ..., X_n = i_n) = \begin{cases} p & j = i_n + 1 \\ 1-p & j = i_n - 1 \\ 0 & else \end{cases}$$

where $|i_1 - i| = 1, ..., |i_n - i_{n-1}| = 1$ and $\begin{matrix} 0 < i_1 < N \\ \vdots \\ 0 < i_n < N \end{matrix}$

Recall: $P(A \cap B) = P(A \mid B) P(B)$

$P(A \cap B \cap C) = P(A \mid B \cap C) P(B \mid C) P(C)$

$P(X_1 = i_1, ..., X_n = i_n) = p^k (1-p)^{n-k}$ and $k = \# j$ such that $i_j = i_{j-1} + 1$

Questions:

(a) $\underbrace{P(X_n = 0 \; eventually)}_{ruin} + \underbrace{P(X_n = N \; eventuallly)}_{success} = 1$?

Equivalently, is $P(0 < X_n < N \;\forall n) = 0$? The answer is yes.

(b) Then if $T = \min\{n \mid X_n = 0 \; or \; N\}$, $P(T < \infty) = 1$, is $ET < \infty$? The answer is yes

(c) What is ET? Answer if $p = \frac{1}{2}$ is $ET = i(N - i)$

(d) What is the distribution of X_T? i.e. What is $P(X_T = N)$ and $P(X_T = 0)$?

Answer: $P(X_T = N) = \frac{i}{N}$; $P(X_T = 0) = \frac{N-i}{N}$

Example: Queueing model

Let Y_k be iid, non-negative, integer valued random variables

$Y_k =$ # arrivals at queue at time k.

X_n = size of queue at time n.

Suppose at each n, 1 person is served. Then,

$X_n = (X_{n-1} - 1)^+$

where $(x)^+ = \begin{cases} x & if \ x \ge 0 \\ 0 & if \ x < 0 \end{cases}$

(a) Is this "stable"? Answer: It is stable iff $EY_k < 1$

(b) If so (i.e. if $EY_k < 1$), what is $\lim_{n\to\infty} P(X_n = k)$?

Galton-Watson Branching Process (Population Growth)

X_n = size of population at time n. Let $X_0 = 1$

$p_i =$ probability a given individual will have i offspring

$p_i \ge 0, \sum_{i=0}^{\infty} p_i = 1$

Conditional on $X_1, ..., X_n$, we have that

$X_{n+1} \stackrel{d}{=} \sum_{i=1}^{X_n} Z_i$, where $P(Z_i = j) = p_j$, Z_i iid

Z_i =# offspring of the ith person in the nth generation

Assume $p_0 + p_1 < 1$ (there is some probability of having two or more offspring

(a) What is $P(X_n = 0 \ eventually)$?

Is $P(X_n = 0 \ eventually) = 1$ or $P(X_n = 0 \ eventually) < 1$?

Answer: $P(X_n = 0 \ eventually) = 1$ if $\sum_{j=0}^{\infty} jp_j \le 1$

$P(X_n = 0 \ eventually) < 1$ if $\sum_{j=0}^{\infty} jp_j > 1$

(b) $P(X_n = 0 \ eventually) + P(\lim_{n\to\infty} X_n = \infty) = 1$? Answer: Yes.

(c) If $\mu = \sum_{j=0}^{\infty} jp_j > 1$, how does X_n behave on $\{\lim_{n\to\infty} X_n = \infty\}$?

Answer: $\lim_{n\to\infty} \frac{X_n}{\mu^n} = w \ge 0$ where $EX_n = \mu^n$. $w > 0$ on $\{X_n \to \infty\}$ if

$\sum_{j=0}^{\infty} j^2 p_j < \infty$