Econ201C: General Equilibrium and Welfare Economics

Problem Set 7

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1 Truthtelling for a costless public good

Suppose $z \in \{0,1\}$ is a binary choice public good that is costless to provide and $v_i(0) = 0$ for all $i = 1, ..., n$. Hence v_i can be identified with the value $v_i(1)$, the amount by which i prefers 1 to 0 if $v_i > 0$ or the amount by which i prefers 0 to 1 if $v_i < 0$.

1.1 Part (a)

Assuming $\mathbf{v} = (v_i) \in V = \times_i V_i$ are the actual preferences of the population, what is the Pareto efficient mapping $z(\mathbf{v})$ from V to $\{0,1\}$ (assuming quasilinearity). (What is the objective required to achieve efficiency?)

1.1.1 Answer

The function $z(\mathbf{v}) = \begin{cases} 1 & \sum_i v_i(1) > 0 \\ 0 & \sum_i v_i(1) \leq 0 \end{cases}$ is Pareto efficient.

1.2 Part (b)

Let $m(\mathbf{v}) = (m_i(\mathbf{v}))$ be the money payments used to induce truthful revelation. Explain (give example) why setting $m_i(\mathbf{v}) \equiv 0$ for all i will not work.

1.2.1 Answer

Suppose there are two individuals 1 and 2 with preferences $v_1(1) = -1$ and $v_2(1) = 2$. Suppose 1 tells the truth. Then he gets.

$$\begin{aligned} u_1(\mathbf{v}|v_1; v_1) &= v_1(z(\mathbf{v})) + m_i(\mathbf{v}) \\ &= v_1(1) + 0 \\ &= -1 \end{aligned}$$

Next, suppose 1 lies and announces $v_1'(1) = 3$. Then,

$$\begin{aligned} u_1(\mathbf{v}|v_1'; v_1) &= v_1(z(\mathbf{v}|v_1')) + m_i(\mathbf{v}|v_1') \\ &= v_1(0) + 0 \\ &= 0 \end{aligned}$$

Therefore, we have that $u_1(\mathbf{v}|v_1'; v_1) > u_1(\mathbf{v}|v_1'; v_1)$. In other words, truth-telling is not a dominant strategy.

1.3 Part (c)

Suppose $h_i(\mathbf{v}_{-i})$ are the "utility rights" granted to $\mathbf{v}_{-i}$. The externalities that i creates in reporting any v_i (which may or not be the truth) are therefore $\sum_{j \neq i} v_j(z(\mathbf{v}|v_i')) - h_i(\mathbf{v}_{-i})$. Demonstrate that if $m_i(\mathbf{v})$ always has to pay the externalities that i creates, then no matter what is $\mathbf{v}_{-i}$, it pays to tell the truth if $z(\mathbf{v})$ satisfies (a), i.e., truth-telling is a dominant strategy.

1.3.1 Answer

Let $\mathbf{v}$ be arbitrary. In this situation, we have:

$$\begin{aligned} u_i(\mathbf{v}|v_i; v_i) &= v_i(z(\mathbf{v})) + m_i(\mathbf{v}) \\ &= v_i(z(\mathbf{v})) + \sum_{j \neq i} v_j(z(\mathbf{v})) - h_i(\mathbf{v}_{-i}) \\ &= g(\mathbf{v}) - h_i(\mathbf{v}_{-i}) \end{aligned}$$

And

$$\begin{aligned} u_i(\mathbf{v}|v_i'; v_i) &= v_i(z(\mathbf{v}|v_i')) + m_i(\mathbf{v}|v_i') \\ &= v_i(z(\mathbf{v}|v_i')) + \sum_{j \neq i} v_j(z(\mathbf{v}|v_i')) - h_i(\mathbf{v}_{-i}) \\ &= g(\mathbf{v}|v_i'; v_i) - h_i(\mathbf{v}_{-i}) \end{aligned}$$

But by definition, we know that $g(\mathbf{v}) \geq g(\mathbf{v}|v_i'; v_i)$. (Since the maximum *total* gains are always greater if everyone is telling the truth.)

Therefore, since $\mathbf{v}$ was arbitrary, it follows that

$$u_i(\mathbf{v}|v_i; v_i) \geq u_i(\mathbf{v}|v_i'; v_i) \;\; \forall v_i' \in V_i \; \forall \mathbf{v}$$

That is, truth-telling is a dominant strategy.

1.4 Part (d)

Suppose $h_i(\mathbf{v}_{-i}) = g(\mathbf{v}_{-i})$, the maximum total gains achievable by the population without i. Call $m_i^{FA}(\mathbf{v})$ the money payment using the principle in (c) after substituting $h_i(\mathbf{v}_{-i}) = g(\mathbf{v}_{-i})$. Show that each $m_i^{FA}(\mathbf{v})$ in (c) is ≤ 0. When is $\sum_i m_i^{FA}(\mathbf{v}) = 0$?

1.4.1 Answer

It is actually quite obvious that $\forall i$, $m_i^{FA}(\mathbf{v}) = \sum_{j \neq i} v_j(z(\mathbf{v})) - g(\mathbf{v}_{-i}) \leq 0$ if you recognize that this is equivalent to $\sum_{j \neq i} v_j(z(\mathbf{v})) \leq g(\mathbf{v}_{-i}) = \sup_z \left\{ \sum_{j \neq i} v_j(z) \right\}$. Obviously, the sum of non-positive number is zero if and only if each number is zero. That is, if $\forall i$,

$$\sum_{j \neq i} v_j(z(\mathbf{v})) - g(\mathbf{v}_{-i}) = 0$$

$$\sum_{j \neq i} v_j(z(\mathbf{v})) = g(\mathbf{v}_{-i}) = \sup_z \left\{ \sum_{j \neq i} v_j(z) \right\}$$

Which occurs only if $\forall i \;\; z(\mathbf{v}) = z(\mathbf{v}_{-i})$, that is, no one is "pivotal" in the sense that

$$sgn\left(\sum_{j \neq i} v_j(1) \right) sgn\left(\sum_{i=1}^{n} v_i(1) \right) \geq 0.$$

1.5 Part (e)

Let $V_i = [-1, 1]$. Suppose n is large, or approaching ∞. What is the "average" value of $m_i^{FA}(\mathbf{v})$ over all populations under the hypothesis in (d)? (What assumptions are you making about the likelihood of any population?)

1.5.1 Answer

1.6 Part (f)

With respect to (d) and (e), show that no matter how large is n there exist $\mathbf{v}$ such that the average value of $m_i^{FA}(\mathbf{v}) < 0$, indicating that on average individuals are causing externalities.-

1.6.1 Answer

1.7 Part (g)

The money payment $m_i^{FA}(\mathbf{v})$ is constructed so that i internalizes any externality that i creates making the last part of (f) appear to be contradictory. Why isn't it a contradiction to say that the sum of money payments $\sum_i m_i^{FA}(\mathbf{v})$ reflect the actual externalities in the population?

1.7.1 Answer

2 Truthtelling with Private Goods

In a double auction model, individuals are divided into buyers and sellers, with $v_b = \beta_b$, the maximum value b is willing to pay to purchase an object and $v_s = \sigma_s$, the minimum price s is willing to accept to sell an object. A population of buyers and sellers is $v = \left(v_{b_j}, v_{s_k}\right) \in V = \times_j V_{b_j} \times_k V_{s_k}$.

2.1 Part (a)

Assuming $\mathbf{v}$ are the actual preferences of the population, what is the Pareto efficient mapping $z(\mathbf{v}) = (z_j(\mathbf{v}), z_k(\mathbf{v}))$ from V to $\times_j D_{b_j} \times_k D_{s_k}$, where $D_{b_j} = \{0, 1\}$ and $D_{s_k} = \{0, -1\}$, i.e., which buyers should be given objects and which sellers should supply objects. (What is the objective required to achieve efficiency?)

2.1.1 Answer

2.2 Part (b)

Let $z^W(\mathbf{v}) = \left(z_i^W(\mathbf{v})\right)$, $i = j, k$, and $p(\mathbf{v})$ be a price-taking equilibrium for $\mathbf{v}$. How does $z^W(\mathbf{v})$ compare with your answer to (a). Explain (give example) why the money payments in a price-taking equilibrium, $m_i(\mathbf{v}) = -p(\mathbf{v}) z_i(\mathbf{v})$ for all $i = j, k$ do not elicit truthful revelation.

2.2.1 Answer

2.3 Part (c)

Suppose $h_i(\mathbf{v}_{-i})$ are the "utility rights" granted to $\mathbf{v}_{-i}$. The externalities that i creates in reporting any v_i (which may or may not be the truth) are therefore $\sum_{l \neq i} v_l(z_l(\mathbf{v}|v_i)) - h_i(\mathbf{v}_{-i})$. Demonstrate that if $m_i(\mathbf{v})$ always has to pay the externalities he creates, then no matter what is $\mathbf{v}_{-i}$, it pays to tell the truth, i.e., truth-telling is a dominant strategy.

2.3.1 Answer

2.4 Part (d)

Suppose $h_i(\mathbf{v}_{-i}) = g(\mathbf{v}_{-i})$, the maximum total gains achievable by the population without i. Call $m_i^{FA}(\mathbf{v})$ the money payment using the principle in (c) after substituting $h_i(\mathbf{v}_{-i}) = g(\mathbf{v}_{-i})$. Show that $m_i^{FA}(\mathbf{v}) \geq -p(\mathbf{v}) z_i^W(\mathbf{v})$ for all $i = j, k$.

2.4.1 Answer

2.5 Part (d')

Show that there is a price-taking equilibrium price $\underline{p}(\mathbf{v})$ satisfying $m_j^{FA}(\mathbf{v}) = -\underline{p}(\mathbf{v}) z_j^W(\mathbf{v})$ for all the buyers, and a price taking equilibrium price $\overline{p}(\mathbf{v})$ satisfying $m_k^{FA}(\mathbf{v}) = -\overline{p}(\mathbf{v}) z_k^W(\mathbf{v})$ for all the sellers. (Is this feature of the double auction model shared by other models?)

2.5.1 Answer

2.6 Part (e)

Let $V_i = [0,1]$ for all $i = j,k$. Suppose n (the number of buyers and sellers) is large, or approaching ∞. What is the "average" value of $m_i^{FA}(\mathbf{v}) - \left[-p(\mathbf{v}) z_i^W(\mathbf{v})\right]$ over all buyers and sellers? (What assumptions are you making about the likelihood of any population?)

2.6.1 Answer

2.7 Part (f)

With respect to (e), show that no matter how large is n there exist $\mathbf{v}$ such that the average value of $m_i^{FA}(\mathbf{v}) - \left[-p(\mathbf{v}) z_i^W(\mathbf{v})\right]$ is positive, indicating that on average individuals are causing externalities.

2.7.1 Answer

2.8 Part (g)

The money payment $m_i^{FA}(\mathbf{v})$ is constructed so that i internalizes any externality that i creates making the last part of (f) appear to be contradictory. Why isn't it a contradiction to say that the sum of money payments $\sum_i m_i^{FA}(\mathbf{v})$ reflect the actual externalities in the population?

2.8.1 Answer

3 Is There a Difference?

Given the similar conclusions in problems 1 and 2, is there any difference between public and private goods?