Econ201C: General Equilibrium and Welfare Economics

Problem Set 6

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1 Externalities in Games

Suppose $v_1(z_1, z_2)$ and $v_2(z_1, z_2)$ are the payoff functions in a normal form game in which each $z_i \in [0, \bar{x}]$ so there is a continuum of strategies. Assume each v_i is concave and differentiable.

1.1 Part (a)

Define a Pareto-efficient play of the game. Explain why such a play can be characterized as maximizing the weighted sum $\lambda_1 v_1(z_1, z_2) + \lambda_2 v_2(z_1, z_2)$ among all possible plays of the game, where λ_i are non-negative and sum to 1. Why is concavity of v_i important? Do you need to assume quasilinearity to justify your answer?

1.1.1 Answer

First, I will start with several preliminary definitions. I apologize for any differences in notation I may introduce.

Definition 1 *A game G is a triple (I, A, π) of players, actions, and payoffs.*

Definition 2 *A play of the game G is a strategy profile $(a_i) \in A = A_1 \times ... \times A_n$.*

Definition 3 *The payoff set associated with the game G is the set*

$$\Pi(G) = \{(\pi_i) : \exists (a_i) \in A_i \text{ satisfying } \pi_i = \pi_i((a_i))\}$$

Now, if we consider the game $G = \left(\{1, 2\}, [0, \bar{x}]^2, \{v_1, v_2\}\right)$ described above, the play $z^* \in [0, \bar{x}]^2$ is Pareto-efficient if

$$\Pi(G) \cap \left\{(v_1(z^*), v_2(z^*)) + \mathbb{R}^2_+ \setminus \{0, 0\}\right\} = \emptyset$$

More generally,

Definition 4 *Let 0 denote the origin and $|I| = n$. The play $z \in A$ is Pareto-efficient if*

$$\Pi(G) \cap \left\{(\pi_i) + \mathbb{R}^n_+ \setminus (0)\right\}$$

That is, no one can be made better off without making someone worse off. By the concavity of the v_i (and convexity of each $A_i = [0, \bar{x}]$), the payoff set associated with G is a convex set. Further, we know that for any play z, the set $\left\{(v_1(z), v_2(z)) + \mathbb{R}^2_+ \setminus \{0, 0\}\right\}$ is convex. By the separating hyperplane theorem, we can always find a hyperplane which separates two disjoint convex sets. That is, for any Pareto-efficient allocation, we can find a λ_1, λ_2 (satisfying $\lambda_i \geq 0$, $\lambda_1 + \lambda_2 = 1$) for which the line $\lambda_1 v_1(z^*) + \lambda_2 v_2(z^*) = k$ is such a separating hyperplane.

Quasilinearity was not necessary for any of this. The assumption of quasilinearity ensures that the Pareto frontier

$$P(G) = \left\{(v_1, v_2) : \Pi(G) \cap \left\{(v_1, v_2) + \mathbb{R}^2_+ \setminus \{0, 0\} = \emptyset\right\}\right\}$$

is "flat."

1.2 Part (b)

What are the FOC for a Pareto efficient play of the game [call it $z^o = (z_1^o, z_2^o)$]? If a play of the game satisfies the FOC for efficiency, why does that imply that it is efficient?

1.2.1 Answer

Since in part (a), we showed we could find a Pareto efficient play by maximizing $V(z) = \lambda_1 v_1(z) + \lambda_2 v_2(z)$ for some λ_1, λ_2 (satisfying $\lambda_i \geq 0$, $\lambda_1 + \lambda_2 = 1$), we have that FOC for a Pareto efficient play of the game are

$$\begin{aligned}(z_1) &: \frac{\partial V(z)}{\partial z_1} = \lambda_1 \frac{\partial v_1(z_1, z_2)}{\partial z_1} + \lambda_2 \frac{\partial v_2(z_1, z_2)}{\partial z_1} = 0 \\ (z_2) &: \frac{\partial V(z)}{\partial z_2} = \lambda_1 \frac{\partial v_1(z_1, z_2)}{\partial z_2} + \lambda_2 \frac{\partial v_2(z_1, z_2)}{\partial z_2} = 0\end{aligned}$$

This implies that any $z^o = (z_1^o, z_2^o)$ satisfying these conditions is efficient because all spillover effects are properly taken into account through the cross derivatives.

1.3 Part (c)

What are the first order conditions for a Nash equilibrium play of the game [call it $z^e = (z_1^e, z_2^e)$]? Explain why λ_i weights play no role. If a play of the game satisfies the FOC for a Nash equilibrium, why does that imply it is a Nash equilibrium?

1.3.1 Answer

The first order conditions for a Nash equilibrium play of the game are:

$$\begin{aligned}(z_1) &: \lambda_1 \frac{\partial v_1(z_1, z_2)}{\partial z_1} = 0 \\ (z_2) &: \lambda_2 \frac{\partial v_2(z_1, z_2)}{\partial z_2} = 0\end{aligned}$$

Or equivalently,

$$\begin{aligned}\frac{\partial v_1(z_1, z_2)}{\partial z_1} &= 0 \\ \frac{\partial v_2(z_1, z_2)}{\partial z_2} &= 0\end{aligned}$$

Suppose $z^e = (z_1^e, z_2^e)$ satisfies the above FOCs. Then we have that:

$$\begin{aligned}v_1(z_1^e, z_2^e) &\geq v_1(z_1', z_2^e) \ \forall z_1' \in [0, \bar{x}] \\ v_2(z_1^e, z_2^e) &\geq v_2(z_1^e, z_2') \ \forall z_2' \in [0, \bar{x}]\end{aligned}$$

Which is the definition of a Nash equilibrium.

1.4 Part (d)

Suppose

$$\begin{aligned}v_1(z_1, z_2) &= A_{11} z_1 - B_{11} z_1^2/2 + A_{12} z_2 - B_{12} z_2^2/2 \\ v_2(z_1, z_2) &= A_{22} z_2 - B_{22} z_2^2/2 + A_{21} z_1 - B_{21} z_1^2/2\end{aligned}$$

where the constraints A_{ij} and B_{ij} are positive.

Find z^o and z^e as functions of the constraints. [Assume $\bar{x}$ is sufficiently large that it is not a binding constraint. Also assume that the weights on each person used to calculate z^o are the same and equal to 1.]

1.4.1 Answer

In order to find z^o, let

$$\begin{aligned} V(z) &= v_1(z_1, z_2) + v_2(z_1, z_2) \\ &= A_{11}z_1 - B_{11}\frac{z_1^2}{2} + A_{12}z_2 - B_{12}\frac{z_2^2}{2} + A_{22}z_2 - B_{22}\frac{z_2^2}{2} + A_{21}z_1 - B_{21}\frac{z_1^2}{2} \end{aligned}$$

Taking FOCs,

$$\begin{aligned} (z_1) &: A_{11} - B_{11}z_1 + A_{21} - B_{21}z_1 = 0 \\ (z_2) &: A_{12} - B_{12}z_2 + A_{22} - B_{22}z_2 = 0 \end{aligned}$$

Solving for the optimal $z^o = (z_1^o, z_2^o)$,

$$\begin{aligned} A_{11} + A_{21} - z_1(B_{11} + B_{21}) &= 0 \\ z_1^o &= \frac{A_{11} + A_{21}}{B_{11} + B_{21}} \\ A_{12} + A_{22} - z_2(B_{12} + B_{22}) &= 0 \\ z_2^o &= \frac{A_{12} + A_{22}}{B_{12} + B_{22}} \end{aligned}$$

Thus, $z^o = \left(\frac{A_{11}+A_{21}}{B_{11}+B_{21}}, \frac{A_{12}+A_{22}}{B_{12}+B_{22}}\right)$.
Taking FOCs to find z^e,

$$\begin{aligned} (z_1) &: A_{11} - B_{11}z_1 = 0 \\ (z_2) &: A_{22} - B_{22}z_2 = 0 \end{aligned}$$

Thus, $z^e = \left(\frac{A_{11}}{B_{11}}, \frac{A_{22}}{B_{22}}\right)$

1.5 Part (e)

Find conditions such that (i) z_1 is undersupplied in equilibrium $[z_1^e < z_1^o]$, (ii) z_1 is oversupplied in equilibrium $[z_1^e > z_1^o]$ and (iii) z_1 is efficiently supplied in equilibrium $[z_1^e = z_1^o]$.

1.5.1 Answer

(i)

$$\begin{aligned} z_1^e &< z_1^o \\ \frac{A_{11}}{B_{11}} &< \frac{A_{11} + A_{21}}{B_{11} + B_{21}} \\ A_{11}B_{11} + A_{11}B_{21} &< B_{11}A_{11} + B_{11}A_{21} \\ A_{11}B_{21} &< A_{21}B_{11} \\ \frac{A_{11}}{B_{11}} &< \frac{A_{21}}{B_{21}} \end{aligned}$$

(ii)

$$\begin{aligned} z_1^e &> z_1^o \\ \frac{A_{11}}{B_{11}} &> \frac{A_{21}}{B_{21}} \end{aligned}$$

(*iii*)

$$\begin{aligned} z_1^e &= z_1^o \\ \frac{A_{11}}{B_{11}} &= \frac{A_{21}}{B_{21}} \end{aligned}$$

1.6 Part (*f*)

Use the Pigovian heuristic comparing private and social benefits and costs to explain undersupply and oversupply in (*d*). How can you explain (*iii*)?

1.6.1 Answer

We can interpret $\frac{A_{11}}{B_{11}}$ as the amount of z_1 person 1 wants to consume. Similarly, we can interpret $\frac{A_{ij}}{B_{ij}}$ as the amount of z_j person i wants j to consume. $(i, j \in \{1, 2\})$ Thus, part (*i*) says that person 2 wants person 1 to consume more than person 1 wants to optimally consume. Part (*ii*) says that person 2 wants person 1 to consume less than person 1 wants to consume. Similarly, part (*iii*) says that person 2 wants person 1 to consume exactly the amount person 1 wants to consume.

This can be related to externalities. If there are beneficial externalities created by person 1, then person 2 will want person 1 to consume more than it is optimal for person 1 to consume. Similarly for negative externalities.

2 The Geometry of the Coordination Problem

The directional derivative of f at $z = (z_1, z_2)$ in the direction $d = (d_1, d_2)$ is:

$$Df\,(z; d) = \lim_{t \searrow 0} \frac{f\,(z + td) - f\,(z)}{t}$$

Two facts about the directional derivative are:

- If f is concave, but not necessarily differentiable, the directional derivative is superadditive: $Df\,(z; d + d') \geq Df\,(z; d) + Df\,(z; d')$.
- If f is differentiable, but not necessarily concave, the directional derivative is linear: there is a $p = (p_1, p_2)$ such that $p \cdot d = Df\,(z; d)$ for all d. Hence, $p\,(d + d') = Df\,(z; d) + Df\,(z; d') = Df\,(z; d + d')$.

Problem 1 assumed that each v_i was concave and differentiable. Therefore so is the weighted sum function $v\,(z) = [\lambda_1 v_1 + \lambda_2 v_2]\,(z)$. Throughout the following, assume the weights are equal. Also, for parts $(a) - (d)$, assume v_i is concave but not necessarily differentiable.

2.1 Part (*a*)

For a feasible z, what are the necessary and sufficient conditions on $Dv\,(z; d)$ for it to be Pareto efficient? [Assume $z >> 0$]

2.1.1 Answer

2.2 Part (*b*)

Explain why the conditions $Dv_1\,(z; (1, 0))\,, Dv_1\,(z; (-1, 0))\,, Dv_2\,(z; (0, 1))$ and $Dv_2\,(z; (0, -1))$ all ≤ 0 are necessary and sufficient conditions for z to be a Nash equilibrium.

2.2.1 Answer

2.3 Part (c)

Suppose (I) $Dv_1(z;(\alpha,0)) = Dv(z;(\alpha,0))$ and $Dv_2(z;(0,\alpha)) = Dv(z;(0,\alpha))$ for all α, positive or negative. Suppose, in addition, that (II) z satisfies the conditions in (b) for Nash equilibrium. Explain why the combination of (I) and (II) is the marginalist version of the Pigovian recipe for the elimination of the harmful effects of externalities.

2.3.1 Answer

2.4 Part (d)

Explain/illustrate why the condition in (c) does NOT suffice for efficiency? Explain why condition (c) would suffice if there were differentiability.

2.4.1 Answer

2.5 Part (e)

Suppose

$$\begin{aligned} v_1(z_1,z_2) &= z_2\left(A_1 z_1 - B_1 z_1^2/2\right) - z_2^{1/2}/2 \\ v_2(z_1,z_2) &= z_1\left(A_2 z_2 - B_2 z_2^2/2\right) - z_1^{1/2}/2 \end{aligned}$$

Verify that v_i are not concave. (They are of course differentiable.)

2.5.1 Answer

2.6 Part (f)

Let $A_i = B_i = 1$. Show that there is a Nash equilibrium at $z_1 = z_2 = 1$.

2.6.1 Answer

2.7 Part (g)

Show that at this Nash equilibrium $\frac{\partial v(z)}{\partial z_i} = 0$. Therefore, it satisfies the differentiable version of (c).

2.7.1 Answer

2.8 Part (h)

Show that the failure of concavity is consistent with the possibility that there are coordiation problems. Find $(\hat{z}_1, \hat{z}_2)$ that are Pareto improvements on $z_1 = z_2 = 1$.

2.8.1 Answer