Econ201C: General Equilibrium and Welfare Economics

Problem Set 5

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1 Public Goods and Pricing in Games

[Background: One of the "disconnects" between game theory and economics is the centrality of prices in the latter and its apparent irrelevance in the former. This problem is intended to build a bridge. It illustrates that the non-cooperative equilibrium concept of correlated equilibrium may be regarded as a variant of Lindahl equilibrium.]

Consider the following game from Myerson, chapter 6.

	b_1	b_2
a_1	$5,1$	$0,0$
a_2	$4,4$	$1,5$

Define the utility function for (row) player 1 and (column) player 2 as:

$$\begin{aligned} v_1 &= \left(v_{11}^1, v_{12}^1, v_{21}^1, v_{22}^1\right) = (5,0,4,1) \\ v_2 &= \left(v_{11}^2, v_{12}^2, v_{21}^2, v_{22}^2\right) = (1,0,4,5) \end{aligned}$$

Define $\bar{X}$ as the set of $x = (x_{11}, x_{12}, x_{21}, x_{22})$ with non-negative elements summing to 1, i.e., randomized plays of the game. The payoff to player i is:

$$v_i(x_i) = \begin{cases} v_i \cdot x_i & \text{if } x_i \in \bar{X} \\ -\infty & \text{otherwise} \end{cases}$$

Let

$$v_1[a_1] = (5,0,5,0) \text{ and } v_1[a_2] = (4,1,4,1)$$

and

$$v_2[b_1] = (1,1,4,4) \text{ and } v_2[b_2] = (0,0,5,5)$$

These vectors are used to describe the payoff to player i who deviates from the actions recommended in $x \in \bar{X}$. For example, $v_1[a_1] \cdot x$ is the expected payoff to 1 from choosing a_1 all the time rather than passively allowing his actions to be determined by x. The deviation gains from any x are therefore based on

$$\begin{aligned} \Delta v_1[a_1] &= v_1[a_1] - v_1 = (0,0,1,-1) \\ \Delta v_1[a_2] &= v_1[a_2] - v_1 = (-1,1,0,0) \\ \Delta v_2[b_1] &= v_2[b_1] - v_2 = (0,1,0,-1) \\ \Delta v_2[b_2] &= v_2[b_2] - v_2 = (-1,0,1,0) \end{aligned}$$

For example, $\Delta v_1[a_1] \cdot x > 0$ means that 1 would gain by deviating from x to choose a_1 all the time.

Definition 1 *A correlated equilibrium for the above game is any* $x \in \bar{X}$ *such that* $\Delta v_1(a_j) \cdot x \leq 0$ *and* $\Delta v_2(b_j) \cdot x \leq 0$ *for* $j \in \{1,2\}$.

The supplier of the public good is player 0. His "technology" of feasible $x_0 = \left(x_{11}^0, x_{12}^0, x_{21}^0, x_{22}^0\right)$ is:

(I) $x_0 \geq 0$; equivalently, $(-1,-1,-1,-1) \cdot x_0 \leq 0$

(II) $\Delta v_1[a_1] \cdot x_0 \leq 0$; $\Delta v_1[a_2] \cdot x_0 \leq 0$; $\Delta v_2[b_1] \cdot x_0 \leq 0$; $\Delta v_2[b_2] \cdot x_0 \leq 0$.

The supplier has the payoff function $v_0 = -c_0$ defined by

$$v_0(x_0) = \begin{cases} 0 & \text{if } x_0 \text{ is feasible,} \\ -\infty & \text{otherwise} \end{cases}$$

In other words, the supply of x_0 is costless, *provided that x_0 satisfies the incentive compatibility constraints in (II)*. Theconstraints in (II) differ from the inequalities defining correlated equilibrium only in that x_0 is not required to be a probability distribution.

Observe that the feasible technology is defined by 5 linear inequalities of the form $y \cdot x_0 \leq 0$, which means that the set of feasible x_0 is a convex cone (constant returns to scale). Facing the prices $p_0 = \left(p_{11}^0, p_{12}^0, p_{21}^0, p_{22}^0\right)$, the supplier's profit function is

$$v_0^*(p_0) = \sup_{x_0} \{v_0 \cdot x_0 - p_0 \cdot x_0\}$$

which equals $-p_0 \cdot x_0$ if x_0 is feasible and $v_0^*(p_0) < \infty$. Because the technology is a cone, $v_0^*(p_0) < \infty$ implies $v_0^*(p_0) = 0$.

1.1 Part (a)

Define Lindahl equilibrium for the above game/economy. I.e., what are the conditions on x_i and p_i, $i \in \{0, 1, 2\}$ constituting a Lindahl equilibrium. In a departure from class, define the price received by the seller as the *negative* of the price paid by buyers. Hence $-p_0 \cdot x_0$ is the profits of the seller. {Note that in this description, individuals 1 and 2 behave as naive price-takers. All the work having to do with incentive compatibility is done by the supplier.]

1.1.1 Answer

Definition 2 *A Lindahl equilibrium (LE) for the above economy is a vector $[x_0, x_1, x_2, p_0, p_1, p_2]$ of allocations and prices satisfying*

1. $v_0^*(p_0) = v_0 \cdot x_0 - p_0 \cdot x_0$
2. $v_1^*(p_1) = \sup_{x_1} \{v_1 \cdot x_1 : p_1 \cdot x_1 = 0\} = v_1 \cdot x_1$
3. $v_2^*(p_2) = \sup_{x_2} \{v_2 \cdot x_2 : p_2 \cdot x_2 = 0\} = v_2 \cdot x_2$
4. $x_0 = x_1 = x_2$
5. $p_0 + p_1 + p_2 = 0$

Note that this definition is general in the sense that the incentive compatibility constraints are not explicit. Rather, they are implicit in the definition of v_0.

1.2 Part (b)

Confirm the claim that in the definition of price-taking (Lindahl) equilibrium for public goods the role of prices and quantities are exactly the reverse of what they are in the definition of price-taking equilibrium for private goods.

1.2.1 Answer

Definition 3 *A price-taking equilibrium (PTE) for the above economy, assuming private goods economy is a vector $[x_0, x_1, x_2, p_0, p_1, p_2]$ of allocations and prices satisfying*

1. $v_0^*(p_0) = v_0 \cdot x_0 - p_0 \cdot x_0$
2. $v_1^*(p_1) = \sup_{x_1} \{v_1 \cdot x_1 : p_1 \cdot x_1 = 0\} = v_1 \cdot x_1$

3. $v_2^*(p_2) = \sup_{x_2} \{v_2 \cdot x_2 : p_2 \cdot x_2 = 0\} = v_2 \cdot x_2$

4. $p_0 = p_1 = p_2$

5. $x_0 + x_1 + x_2 = 0$

Condition (4) states that prices are a public good in the sense that they are equal for all individuals. Condition (5) is the market clearing condition for this economy assuming no outside endowments. Clearly, in such a statement of equilibrium, the roles of prices and quantities are exactly the reverse of what they are in the Lindahl equilibrium from part (a).

1.3 Part (c)

Confirm that the prices

Row Player	b_1	b_2	Column Player	b_1	b_2
a_1	$5/3$	0	a_1	$-7/3$	0
a_2	$2/3$	$-7/3$	a_2	$2/3$	$5/3$

can be used to find a Lindahl equilibrium for $x_0 = (1/3, 0, 1/3, 1/3)$.

Further, confirm that at the equilibrium p_0 some "commodities" have negative prices and some have positive prices and that the supplier chooses to maximize profits by supplying commodities with negative prices. Can you explain why?

1.3.1 Answer

For person 1, we have

$$\begin{aligned}
v_1^*(p_1) &= \max_{x_1 \in \bar{X}} \{v_1 \cdot x_1 : p_1 \cdot x_1 = 0\} \\
&= \max_{x_1 \in \bar{X}} \left\{(5,0,4,1) \cdot \left(x_{11}^1, x_{12}^1, x_{21}^1, x_{22}^1\right) : (5/3, 2/3, -7/3) \cdot \left(x_{11}^1, x_{12}^1, x_{21}^1, x_{22}^1\right) = 0\right\} \\
&= \max_{x_1} \left\{5x_{11}^1 + 4x_{21}^1 + x_{22}^1 : \frac{5}{3}x_{11}^1 + \frac{2}{3}x_{21}^1 - \frac{7}{3}x_{22}^1 = 0, x_{11}^1 + x_{12}^1 + x_{21}^1 + x_{22}^1 = 1\right\}
\end{aligned}$$

Since x_{12}^1 neither contributes any utility to 1 nor provides any financial benefit, it is obvious that $x_{12}^1 = 0$

$$\begin{aligned}
v_1^*(p_1) &= \max_{x_1} \left\{5x_{11}^1 + 4x_{21}^1 + x_{22}^1 : \frac{5}{3}x_{11}^1 + \frac{2}{3}x_{21}^1 - \frac{7}{3}x_{22}^1 = 0, x_{11}^1 + x_{21}^1 + x_{22}^1 = 1\right\} \\
&= \max_{x_1} \left\{5\left(1 - x_{21}^1 - x_{22}^1\right) + 4x_{21}^1 + x_{22}^1 : \frac{5}{3}\left(1 - x_{21}^1 - x_{22}^1\right) + \frac{2}{3}x_{21}^1 - \frac{7}{3}x_{22}^1 = 0\right\} \\
&= \max_{x_1} \left\{5 - x_{21}^1 - 4x_{22}^1 : x_{21}^1 = \frac{5 - 12x_{22}^1}{3}\right\} \\
&= \max_{x_1} \left\{5 - \frac{5 - 12x_{22}^1}{3} - 4x_{22}^1\right\} \\
&= \max_{x_1} \left\{\frac{10}{3}\right\} = \frac{10}{3}
\end{aligned}$$

Note that

$$v_1 \cdot x_1 = (5,0,4,1) \cdot \left(\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}\right) = \frac{10}{3}$$

Therefore, we have that $\left(\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}\right) \in \arg\max v_1^*(p_1)$. (Further, it is obvious that $\left(\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}\right)$ satisfies the budget constraint and is a probability vector). Therefore, we have that person 1 would optimally choose the vector $\left(\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}\right)$

For person 2, we have (making the same argument as above regarding x_{12}^2)

$$\begin{aligned}
v_2^*(p_2) &= \max_{x_2}\left\{x_{11}^2+4x_{21}^2+5x_{22}^2 : -\frac{7}{3}x_{11}^2+\frac{2}{3}x_{21}^2+\frac{5}{3}x_{22}^2=0, x_{11}^2+x_{21}^2+x_{22}^2=1\right\} \\
&= \max_{x_2}\left\{\left(1-x_{21}^2-x_{22}^2\right)+4x_{21}^2+5x_{22}^2 : -\frac{7}{3}\left(1-x_{21}^2-x_{22}^2\right)+\frac{2}{3}x_{21}^2+\frac{5}{3}x_{22}^2=0\right\} \\
&= \max_{x_2}\left\{1-3x_{21}^2-4x_{22}^2 : x_{21}^2=x_{21}^2=\frac{7-12x_{22}^2}{9}\right\} \\
&= \max_{x_2}\left\{1+\frac{36x_{22}^2-21}{9}-4x_{22}^2 : x_{21}^2=\frac{7-12x_{22}^2}{9}\right\} \\
&= \max_{x_2}\left\{\frac{10}{3}\right\}=\frac{10}{3}
\end{aligned}$$

Note that

$$v_2\cdot x_2=(1,0,4,5)\cdot\left(\frac{1}{3},0,\frac{1}{3},\frac{1}{3}\right)=\frac{10}{3}$$

Therefore, we have that $\left(\frac{1}{3},0,\frac{1}{3},\frac{1}{3}\right)\in\arg\max v_2^*(p_2)$. (Further, it is obvious that $\left(\frac{1}{3},0,\frac{1}{3},\frac{1}{3}\right)$ satisfies the budget constraint and is a probability vector). Therefore, we have that person 2 would optimally choose the vector $\left(\frac{1}{3},0,\frac{1}{3},\frac{1}{3}\right)$.

Since $p_0+p_1+p_2=0$, we have that

$$\begin{aligned}
p_0 &= -(p_1+p_2) \\
&= -((5/3,0,2/3,-7/3)+(-7/3,0,2/3,5/3)) \\
&= -((-2/3,0,4/3,-2/3)) \\
&= (2/3,0,-4/3,2/3)
\end{aligned}$$

Because the production set for person 0 is a convex cone, it must be that $v_0^*(p_0)=0$ and x_0 is feasible. First, I will check that indeed, $x_0=(1/3,0,1/3,1/3)$ is feasible:

$$\begin{aligned}
(-1,0,0,0)\cdot x_0 &= (-1,0,0,0)\cdot(1/3,0,1/3,1/3)=-\frac{1}{3}\le 0 \\
(0,-1,0,0)\cdot x_0 &= (0,-1,0,0)\cdot(1/3,0,1/3,1/3)=0\le 0 \\
(0,0,-1,0)\cdot x_0 &= (0,0,-1,0)\cdot(1/3,0,1/3,1/3)=-\frac{1}{3}\le 0 \\
(0,0,0,-1)\cdot x_0 &= (0,0,0,-1)\cdot(1/3,0,1/3,1/3)=-\frac{1}{3}\le 0 \\
\Delta v_1[a_1]\cdot x_0 &= (0,0,1,-1)\cdot(1/3,0,1/3,1/3)=0\le 0 \\
\Delta v_1[a_2]\cdot x_0 &= (-1,1,0,0)\cdot(1/3,0,1/3,1/3)=-\frac{1}{3}\le 0 \\
\Delta v_2[b_1]\cdot x_0 &= (0,1,0,-1)\cdot(1/3,0,1/3,1/3)=-\frac{1}{3}\le 0 \\
\Delta v_2[b_2]\cdot x_0 &= (-1,0,1,0)\cdot(1/3,0,1/3,1/3)=0\le 0
\end{aligned}$$

Therefore, x_0 is feasible. Furthermore, the profits are zero

$$\begin{aligned}
v_0^*(p_0) &= -p_0\cdot x_0 \\
&= (2/3,0,-4/3,2/3)\cdot(1/3,0,1/3,1/3) \\
&= \frac{2}{9}-\frac{4}{9}+\frac{2}{9}=0
\end{aligned}$$

The intuition that the producer must provide some of the negatively priced commodity is that, due to the incentive compatibility constraints, in order to provide some of the commodities with positive prices, the supplier must also provide some of the commodities with negative prices.

1.4 Part (d)

Suppose the incentive constraints in (II) were eliminated. Find a Lindahl equilibrium that maximizes the sum of utilities among all feasible plays of the game (and is therefore efficient).

1.4.1 Answer

Since we are ignoring the incentive constraints, we can effectively ignore the producer (except to note that profits must be zero), since the function of the producer is to impose these constraints. Keeping this in mind, we want to

$$\begin{aligned}
& \max_{x \in \bar{X}} \{v_1(x) + v_2(x)\} \\
= & \max_{x \in \bar{X}} \{v_1 \cdot x + v_2 \cdot x\} \\
= & \max_{x \in \bar{X}} \{(5,0,4,1) \cdot (x_{11}, x_{12}, x_{21}, x_{22}) + (1,0,4,5) \cdot (x_{11}, x_{12}, x_{21}, x_{22})\} \\
= & \max_{x \in \bar{X}} \{5x_{11} + x_{11} + 4x_{21} + 4x_{21} + x_{22} + 5x_{22}\} \\
= & \max_{x \in \bar{X}} \{6x_{11} + 8x_{21} + 6x_{22}\}
\end{aligned}$$

Clearly, since the social marginal benefit of $x_{21} = 8$ which is greater than the social marginal benefits of $x_{11}, x_{12},$ and x_{22}, we have that $x = (0,0,1,0)$ is the optimal vector. If we have that

$$\begin{aligned}
p_1 &= (1,0,0,-2) \\
p_2 &= (-2,0,0,1)
\end{aligned}$$

Then $x = (0,0,1,0)$ would be optimal for both 1 and 2 to pick. Also, profits would clearly be zero since

$$\begin{aligned}
p_0 &= -(p_1 + p_2) \\
&= (1,0,0,1) \\
v_0^*(p_0) &= -p_0 \cdot x = (-1,0,0,-1) \cdot (0,0,1,0) = 0
\end{aligned}$$