

Problem Set 4

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1 Question 1

[Background: An elementary example of a bargaining problem has one buyer and one seller exchanging one unit of an indivisible commodity. If the buyer's valuation is 1 and the seller's valuation is 0, an equilibrium price is any $p \in [0, 1]$. Hence, the buyer receives the surplus $1 - p$ and the seller receives the surplus p .]

An ultimatum game can be regarded as a two-person exchange model with a procedure attached in which, for example, b gets to propose any division of the total surplus 1 and s can either agree or disagree. In the latter case, both get nothing.

1.1 Part (a)

Define b 's *MP relative to the procedure* as the difference between what he would get in the unique subgame perfect Nash equilibrium of the game when b is the proposer compared to that when b is the responder. Define s 's *MP relative to the procedure* as 0 since s has already lost the possibility to propose in this procedure. Confirm that both individuals get their *MP*'s

1.1.1 Answer

I will begin this part of the response with several notational definitions which I will follow throughout the problem.

Definition 1 Call $P_{i_1 i_2 \dots i_n}^n$ the procedure of length n in which $i_j \in \{b, s\}$ makes the j th offer (and hence $\{b, s\} \setminus i_j$ decides whether or not to accept).

Definition 2 Let $SPNE(P_{i_1 i_2 \dots i_n}^n)$ denote the subgame perfect Nash equilibrium of the bargaining game with respect to the procedure $P_{i_1 i_2 \dots i_n}^n$.

Definition 3 Let $\pi_i(SPNE(P_{i_1 i_2 \dots i_n}^n))$ be the payoff to $i \in \{b, s\}$ under the subgame perfect Nash equilibrium of the bargaining game with respect to the procedure $P_{i_1 i_2 \dots i_n}^n$.

Definition 4 $MP_i^t(P_{i_1 \dots i_n}^n) \equiv \begin{cases} \pi_i(SPNE(P_{i_1 \dots i_n}^n)) - \pi_i(SPNE(P_{i_1 \dots \hat{i}_t \dots i_n}^n)) & i_t = i \\ 0 & \text{else} \end{cases}$ where $\hat{i}_t = \{b, s\} \setminus i_t$.

Definition 5 $MP_i(P_{i_1 \dots i_n}^n) = \sum_{t=1}^n MP_i^t(P_{i_1 \dots i_n}^n)$.

Clearly, $(\pi_b(SPNE(P_b^1)), \pi_s(SPNE(P_b^1))) = (1, 0)$.

Further, it is also obvious that $(\pi_b(SPNE(P_s^1)), \pi_s(SPNE(P_s^1))) = (0, 1)$.

Therefore,

$$\begin{aligned} MP_b(P_b^1) &= \pi_b(SPNE(P_b^1)) - \pi_b(SPNE(P_s^1)) = 1 - 0 = 1 \\ MP_s(P_b^1) &= 0 \end{aligned}$$

This leads us to the desired result that

$$(MP_b(P_b^1), MP_s(P_b^1)) = (1, 0) = (\pi_b(SPNE(P_b^1)), \pi_s(SPNE(P_b^1)))$$

1.2 Part (b)

In a two round variant, if s disagrees in the first round, the surplus to be shared in the second round shrinks to $0 < \delta < 1$.

Suppose s is the proposer in the second round. What is the MP to b of being the proposer in the first round, i.e., how much does b gain in the equilibrium payoffs by being the proposer in the first round rather than s , given that s will be the proposer in the second round. What is the MP to s of being the proposer in the second round, i.e., how much does s gain by being the proposer in the second round rather than b , given that b is the proposer in the first round? Show that in the Subgame Perfect Nash equilibrium (SPNE), each player receives his MP relative to the procedure.

1.2.1 Answer

In my notation, this question asks us to compare

$$(MP_b(P_{bs}^2), MP_s(P_{bs}^2))$$

with

$$(\pi_b(SPNE(P_{bs}^2)), \pi_s(SPNE(P_{bs}^2)))$$

This requires the following calculations which by what seemingly appears to be hand-waving, have been calculated using backward induction:

$$\begin{aligned} (\pi_b(SPNE(P_{bs}^2)), \pi_s(SPNE(P_{bs}^2))) &= (1 - \delta, \delta) \\ (\pi_b(SPNE(P_{bb}^2)), \pi_s(SPNE(P_{bb}^2))) &= (1, 0) \\ (\pi_b(SPNE(P_{ss}^2)), \pi_s(SPNE(P_{ss}^2))) &= (0, 1) \end{aligned}$$

Therefore,

$$\begin{aligned} MP_b^1(P_{bs}^2) &= 1 - \delta - 0 = 1 - \delta \\ MP_b^2(P_{bs}^2) &= 0 \\ MP_s^1(P_{bs}^2) &= 0 \\ MP_s^2(P_{bs}^2) &= \delta - 0 = \delta \end{aligned}$$

This gives us

$$\begin{aligned} MP_b(P_{bs}^2) &= 0 + 1 - \delta = 1 - \delta \\ MP_s(P_{bs}^2) &= \delta + 0 = \delta \end{aligned}$$

Which yields the desired result:

$$(MP_b(P_{bs}^2), MP_s(P_{bs}^2)) = (1 - \delta, \delta) = (\pi_b(SPNE(P_{bs}^2)), \pi_s(SPNE(P_{bs}^2)))$$

1.3 Part (c)

In a three round variant, no agreement in the first two rounds reduces the available surplus to δ^2 in the third round. There are 2^3 possible sequences in which b and s may be proposer and responder. Call each such sequence a *procedure*, P .

Define $MP_i^t(P)$ as the marginal product of the right to be the proposer at $t = 1, 2, 3$. Define it as the change in the SPNE payoff that would occur if i is the proposer in P at t but that right were assigned to the other individual instead. If i is not the proposer in P at t , then $MP_i^t(P) = 0$. Define the individual's marginal product relative to the procedure as

$$MP_i(P) = \sum_{t=1}^3 MP_i^t(P).$$

Show that the payoff to $i = b, s$ in the SPNE for P is $MP_i(P)$.

1.3.1 Answer

It is first necessary to calculate the SPNE payoffs for each of the eight procedures. I will do so by backward induction without making my calculations explicit:

$$\begin{aligned}
(\pi_b(SPNE(P_{bbb}^3)), \pi_s(SPNE(P_{bbb}^3))) &= (1, 0) \\
(\pi_b(SPNE(P_{bbs}^3)), \pi_s(SPNE(P_{bbs}^3))) &= (1 - \delta^2, \delta^2) \\
(\pi_b(SPNE(P_{bsb}^3)), \pi_s(SPNE(P_{bsb}^3))) &= (1 - \delta + \delta^2, \delta - \delta^2) \\
(\pi_b(SPNE(P_{bss}^3)), \pi_s(SPNE(P_{bss}^3))) &= (1 - \delta, \delta) \\
(\pi_b(SPNE(P_{sbb}^3)), \pi_s(SPNE(P_{sbb}^3))) &= (\delta, 1 - \delta) \\
(\pi_b(SPNE(P_{sbs}^3)), \pi_s(SPNE(P_{sbs}^3))) &= (\delta - \delta^2, 1 - \delta + \delta^2) \\
(\pi_b(SPNE(P_{ssb}^3)), \pi_s(SPNE(P_{ssb}^3))) &= (\delta^2, 1 - \delta^2) \\
(\pi_b(SPNE(P_{sss}^3)), \pi_s(SPNE(P_{sss}^3))) &= (0, 1)
\end{aligned}$$

This will give us (with some effort):

$$\begin{aligned}
MP_b(P_{bbb}^3) &= [1 - (1 - \delta^2)] + [1 - (1 - \delta + \delta^2)] + [1 - \delta] = 1 = \pi_b(SPNE(P_{bbb}^3)) \\
MP_b(P_{bbs}^3) &= [0] + [1 - \delta^2 - (1 - \delta)] + [1 - \delta^2 - (\delta - \delta^2)] = 1 - \delta^2 = \pi_b(SPNE(P_{bbs}^3)) \\
MP_b(P_{bsb}^3) &= [1 - \delta + \delta^2 - (1 - \delta)] + [0] + [1 - \delta + \delta^2 - \delta^2] \\
&= 1 - \delta + \delta^2 = \pi_b(SPNE(P_{bsb}^3)) \\
MP_b(P_{bss}^3) &= [0] + [0] + [1 - \delta - 0] = 1 - \delta = \pi_b(SPNE(P_{bss}^3)) \\
MP_b(P_{sbb}^3) &= [\delta - (\delta - \delta^2)] + [\delta - \delta^2] + [0] = \delta = \pi_b(SPNE(P_{sbb}^3)) \\
MP_b(P_{sbs}^3) &= [0] + [\delta - \delta^2 - 0] + [0] = \delta - \delta^2 = \pi_b(SPNE(P_{sbs}^3)) \\
MP_b(P_{ssb}^3) &= [\delta^2 - 0] + [0] + [0] = \delta^2 = \pi_b(SPNE(P_{ssb}^3)) \\
MP_b(P_{sss}^3) &= [0] + [0] + [0] = 0 = \pi_b(SPNE(P_{sss}^3))
\end{aligned}$$

And

$$\begin{aligned}
MP_s(P_{bbb}^3) &= [0] + [0] + [0] = 0 = \pi_s(SPNE(P_{bbb}^3)) \\
MP_s(P_{bbs}^3) &= [\delta^2 - 0] + [0] + [0] = \delta^2 = \pi_s(SPNE(P_{bbs}^3)) \\
MP_s(P_{bsb}^3) &= [0] + [\delta - \delta^2 - 0] + [0] = \delta - \delta^2 = \pi_s(SPNE(P_{bsb}^3)) \\
MP_s(P_{bss}^3) &= [\delta - (\delta - \delta^2)] + [\delta - \delta^2] + [0] = \delta = \pi_s(SPNE(P_{bss}^3)) \\
MP_s(P_{sbb}^3) &= [0] + [0] + [1 - \delta - 0] = 1 - \delta = \pi_s(SPNE(P_{sbb}^3)) \\
MP_s(P_{sbs}^3) &= [1 - \delta + \delta^2 - (1 - \delta)] + [0] + [1 - \delta + \delta^2 - (\delta^2)] \\
&= 1 - \delta + \delta^2 = \pi_s(SPNE(P_{sbs}^3)) \\
MP_s(P_{ssb}^3) &= [0] + [1 - \delta^2 - (1 - \delta)] + [1 - \delta^2 - (\delta - \delta^2)] = 1 - \delta^2 = \pi_s(SPNE(P_{ssb}^3)) \\
MP_s(P_{sss}^3) &= [1 - (1 - \delta^2)] + [1 - (1 - \delta + \delta^2)] + [1 - \delta] = 1 = \pi_s(SPNE(P_{sss}^3))
\end{aligned}$$

Therefore, we have that $MP_i(P^3) = \pi_i(SPNE(P^3)) \forall i \in \{b, s\} \forall P^3$.

1.4 Part (d)

With T rounds there are 2^T possible procedures. As $T \rightarrow \infty$, provide an heuristic argument about the set of possible payoffs to b for all possible procedures, i.e., $\cup_P MP_b(P)$, when $\delta > \frac{1}{2}$.

1.4.1 Answer

I am not certain, exactly, what an heuristic argument is, but it appears, though it is difficult to prove, that $\forall \delta \geq \frac{1}{2} \forall x \in [0, 1], \exists P(\delta)$ of infinite length such that

$$(\pi_b(SPNE(P(\delta))), \pi_s(SPNE(P(\delta)))) = (x, 1 - x)$$

Consider the subset of procedures in which there are only finitely many times that bargaining alternates. That is, there exists some T such that for all $t \geq T$, the same player makes all offers. We can partition this subset into two sets: one in which the player who makes the first move makes all the moves beyond this T , and one in which the player who does not make the first move makes all the moves beyond this T . Denote the first such set \mathbb{P}^1 and the latter set \mathbb{P}^2 . It can be proven with little effort that $\mathbb{P}^1 \cup \mathbb{P}^2$ is dense in the set of all procedures of infinite length under any metric.

Let $P_{i_{n_1} j_{n_2} i_{n_3} \dots j_{n_{k-1}} i}^\infty \in \mathbb{P}^1$ denote the procedure in which i makes the first n_1 offers, $j \neq i$ makes the following n_2 offers, i makes the next n_3 offers, ..., j makes the next n_{k-1} offers and i makes all the offers starting from period $t = 1 + \sum_{i=1}^{k-1} n_i$. The equilibrium payoff of this game will be (the proof of this would be a very difficult exercise):

$$\begin{aligned} \pi_i \left(SPNE \left(P_{i_{n_1} j_{n_2} i_{n_3} \dots j_{n_{k-1}} i}^\infty \right) \right) &= \delta^{\sum_{i=1}^{k-1} n_i} - \delta^{\sum_{i=1}^{k-2} n_i} + \delta^{\sum_{i=1}^{k-3} n_i} - \dots + -\delta^{n_1} + 1 \\ \pi_j \left(SPNE \left(P_{i_{n_1} j_{n_2} i_{n_3} \dots j_{n_{k-1}} i}^\infty \right) \right) &= -\delta^{\sum_{i=1}^{k-1} n_i} + \delta^{\sum_{i=1}^{k-2} n_i} - \delta^{\sum_{i=1}^{k-3} n_i} + \dots - \delta^{n_1+n_2} + \delta^{n_1} \end{aligned}$$

Let $P_{i_{n_1} j_{n_2} i_{n_3} \dots i_{n_{k-1}} j}^\infty \in \mathbb{P}^2$ denote the procedure in which i makes the first n_1 offers, $j \neq i$ makes the following n_2 offers, i makes the next n_3 offers, ..., i makes the next n_{k-1} offers, and j makes all the offers starting from period $t = 1 + \sum_{i=1}^{k-1} n_i$. The equilibrium payoff of this game will be (the proof of this would be a very difficult exercise):

$$\begin{aligned} \pi_i \left(SPNE \left(P_{i_{n_1} j_{n_2} i_{n_3} \dots i_{n_{k-1}} j}^\infty \right) \right) &= -\delta^{\sum_{i=1}^{k-1} n_i} + \delta^{\sum_{i=1}^{k-2} n_i} - \delta^{\sum_{i=1}^{k-3} n_i} + \dots - \delta^{n_1} + 1 \\ \pi_j \left(SPNE \left(P_{i_{n_1} j_{n_2} i_{n_3} \dots i_{n_{k-1}} j}^\infty \right) \right) &= \delta^{\sum_{i=1}^{k-1} n_i} - \delta^{\sum_{i=1}^{k-2} n_i} + \delta^{\sum_{i=1}^{k-3} n_i} - \dots + -\delta^{n_1+n_2} + \delta^{n_1} \end{aligned}$$

The question thus becomes: for $\delta \geq \frac{1}{2} \forall x \in [0, 1]$, can we find a Cauchy sequence $\{P(\delta)\} \subset \mathbb{P}^1 \cup \mathbb{P}^2$ with limit $P(\delta)$ such that

$$(\pi_b(SPNE(P(\delta))), \pi_s(SPNE(P(\delta)))) = (x, 1 - x)?$$

Since this information is redundant in a sense, all we need is that $\forall \delta \geq \frac{1}{2}, \forall x \in [0, 1]$ we can find a Cauchy sequence $\{P(\delta)\} \subset \mathbb{P}^1 \cup \mathbb{P}^2$ with limit $P(\delta)$ such that

$$\pi_b(SPNE(P(\delta))) = x$$

Establishing the existence of such a sequence is very difficult. Having made this claim, I will end my heuristic argument with the assertion that this existence proof is indeed possible and hence $\cup_P MP_b(P) = [0, 1]$.

2 Question 2

[Background: The procedure for resolving bargaining problems described above can be interpreted as having "zero transactions cost" in the sense that no matter what the procedure, bargaining is always efficiently resolved. The way the term "zero transactions costs" is usually understood, (1) there are no transactions costs to begin with and there is also an assumption, rarely if ever mentioned, that (2) individuals are not allowed to create them. Assumption (2) therefore places considerable restrictions on individuals' ability to "innovate." It therefore shares the same features as price-taking and game-taking. Nevertheless, even

though the individuals start out in a zero transactions costs environment, we want to allow individuals to *create* transactions costs if it is in their self-interest.]

In the game described above, the proposer uses a costless channel of communication to convey his offer. Assume an alternating offer sequence with (b, s) as the sequence of proposers. Add the following innovation as a first move that takes place prior to the sequence of offers: b can do nothing and the (b, s) sequence is played; or, upon paying a cost of c , b can create noise in the communication channel that s uses to make an offer to b so that s 's offer is delivered only after a one period delay, in effect making the game into $(b, 0, s)$ where 0 stands for the fact that nothing happens in period 2 while s is trying to get in touch with b , except that the surplus continues to shrink. [Note: This is a game of perfect information in which s observes whether or not b has chosen to create "noise."]

2.1 Part (a)

Ignoring the cost c and assuming the game $(b, 0, s)$ is played, is the outcome efficient? Is the outcome efficient after c is taken into account?

2.1.1 Answer

If we assume $c = 0$, then

$$\begin{aligned} (\pi_b (SPNE (P_{b0s}^3)) - c, \pi_s (SPNE (P_{b0s}^3))) &= (\pi_b (SPNE (P_{b0s}^3)), \pi_s (SPNE (P_{b0s}^3))) \\ &= (1 - \delta^2, \delta^2) \end{aligned}$$

In this situation, $\pi_b (SPNE (P_{b0s}^3)) + \pi_s (SPNE (P_{b0s}^3)) = 1 - \delta^2 + \delta^2 = 1$. That is, it is an efficient allocation of resources. If we assume $c > 0$, then we have that

$$(\pi_b (SPNE (P_{b0s}^3)) - c, \pi_s (SPNE (P_{b0s}^3))) = (1 - \delta^2 - c, \delta^2)$$

And hence $\pi_b (SPNE (P_{b0s}^3)) - c + \pi_s (SPNE (P_{b0s}^3)) = 1 - c < 1$, which is obviously not efficient.

2.2 Part (b)

Can you find values of δ and c so that b would want to create noise?

2.2.1 Answer

If it is ever the case that $\pi_b (SPNE (P_{bs}^2)) \leq \pi_b (SPNE (P_{b0s}^3)) - c$, b would want to create noise. Thus, b would want to create noise when

$$\begin{aligned} \pi_b (SPNE (P_{bs}^2)) &\leq \pi_b (SPNE (P_{b0s}^3)) - c \text{ or} \\ 1 - \delta &\leq 1 - \delta^2 - c \\ c &\leq \delta - \delta^2 \\ c &\leq \delta(1 - \delta) \end{aligned}$$

One such parametrization would be $\delta = \frac{1}{2}$ and $c = \frac{1}{5}$. In such a case, b would strictly prefer to create noise and receive an equilibrium payoff of

$$\begin{aligned} 1 - \delta^2 - c &= 1 - \left(\frac{1}{2}\right)^2 - \frac{1}{5} \\ &= 1 - \frac{1}{4} - \frac{1}{5} \\ &= \frac{11}{20} \end{aligned}$$

Whereas if he did not create noise, he would receive an equilibrium payoff of

$$\begin{aligned} 1 - \delta &= 1 - \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

2.3 Part (c)

Zero transactions costs is taken to imply that there are never any efficiency losses due to bargaining problems, or fights over the surplus. One way to justify this conclusion is that zero transactions costs includes an assumption that there is an angel who sprinkles a magical dust over people eliminating the selfish parts of their natures. Can you come up with a better assumption justifying the absence of efficiency losses due to bargaining problems?

2.3.1 Answer

I personally like the assumption of the existence of an angel with magical dust, as this same assumption supports many of the other outrageous results we have acquired in our other courses. (Why didn't I think of that before?)

In all seriousness, though, a couple plausible assumptions we could make to justify the absence of efficiency losses due to bargaining problems are:

1. $c > \delta(1 - \delta)$. That is, it is sufficiently costly to create transactions costs. A possible justification for this includes c being a measure of present value of future negative reputational effects due to creating such transactions costs (people aren't as likely to put themselves into the position of being second mover with you in the future if you burn them today).
2. The existence of outside opportunities which are perfect substitutes. Such existence would align marginal productivity and payoffs, i.e. perfect competition.