Econ201C: General Equilibrium and Welfare Economics

Problem Set 2

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1 Demand, Inverse Demand and Indirect Utility

1.1 Part (a)

Show that the following are equivalent:

1. (Inverse demand) $p \in \partial v(z)$;
2. (Indirect utility) $v^*(p) = v(z) - p \cdot z$;
3. (Demand) $(z, m) \in d(v, p, 0)$, where $d(v, p, 0) = \arg\max\{v(z) + m : pz + m = 0\}$

1.1.1 Answer to (a)

Recall the definition of the subdifferential:

$$\partial v(z) = \{p : v(z) - pz \geq v(z') - pz' \quad \forall z'\}$$

Therefore, $p \in \partial v(z) \iff v(z) - pz \geq v(z') - pz' \quad \forall z'$

Recall the definition of the indirect utility function :

$$\begin{aligned} v^*(p) &= \sup_{z'}\{v(z') + m : pz' + m = 0\} \\ &= \sup_{z'}\{v(z') - pz'\} \end{aligned}$$

Therefore, $v^*(p) = v(z) - pz \iff v(z) - pz = \sup_{z'}\{v(z') - pz'\} \geq v(z') - pz' \quad \forall z'$

Recall the definition of the demand set

$$d(v, p, 0) = \arg\max\{v(z) + m : pz + m = 0\}$$

Therefore, $(z, m) \in d(v, p, 0) \iff (z, m) \in \arg\max\{v(z) + m : pz + m = 0\}$. That is, (z, m) maximizes $v(z) + m$ subject to $pz + m = 0$ or $m = -pz$. Equivalently, $v(z) - pz \geq v(z') - pz' \quad \forall z'$.

Therefore, all three notions are equivalent, since $\iff$ is an equivalence relation.

1.2 Part (b)

Show that for any scalar w, $(z, m) \in d(v, p, w) = \arg\max\{v(z) + m : pz + m = w\}$ if and only if $(z, m, -w) \in d(v, p, 0)$.

1.2.1 Answer to (b)

Let w be a scalar. Suppose $(z,m) \in d(v,p,w)$. Then

$$\begin{aligned}
(z,m) &\in \arg\max\{v(z)+m : pz+m=w\} \\
&\iff (z,w-pz) \in \arg\max\{v(z)+w-pz\} \\
&\iff v(z)+w-pz \geq v(z')+w-pz' \quad \forall z' \\
&\iff v(z)+w-pz-w \geq v(z')-pz'+w-w \quad \forall z' \\
&\iff v(z)-pz \geq v(z')-pz' \quad \forall z' \\
&\iff (z,-pz) \in \arg\max\{v(z)-pz\} \\
&\iff (z,m-w) \in \arg\max\{v(z)+m-w : pz+m-w=0\} \\
&\iff (z,m-w) \in d(v,p,0)
\end{aligned}$$

1.3 Part (c)

Suppose $v=-c(z)$, where $c(z)$ is the minimum amount of the money commodity needed to produce z. Divide c into two cases: (A) $c(z)$ is either 0 or $+\infty$ and (B) $0 \leq c(z) \leq +\infty$. Interpret (1) – (3) in terms of supply and profits.

1.3.1 Answer to (c)

(A) Suppose $c(z) \in \{0 \text{ or } +\infty\}$. Economically, this says that money is neither an input nor an output.

1. The definition of the subdifferential $\partial v(z)$ becomes:

$$\partial[-c](z) = \{p : -pz \geq -pz' \ \forall z',\ c(z)=0\}$$

   Therefore, $p \in \partial v(z) \Rightarrow -pz \geq -pz' \ \forall z'$ and $c(z)=0$. (z is in the production set)

2. The corresponding definition of the profit function is:

$$\begin{aligned}
\pi^*(p) &= \sup_{m,z}\{m-c(z) : pz+m=0\} \\
&= \sup_{z}\{-pz-c(z)\} \\
&= \sup_{z}\{-pz :\ c(z)=0\}
\end{aligned}$$

   Therefore, $\pi^*(p) = -pz-c(z) \Rightarrow -pz \geq -pz' \ \forall z'$ and $c(z)=0$

3. The corresponding definition of the supply correspondence is

$$\begin{aligned}
s(c,p,0) &= \arg\max\{m-c(z) : m+pz=0\} \\
&= \arg\max\{-pz-c(z)\} \\
&= \arg\max\{-pz : c(z)=0\}
\end{aligned}$$

   Therefore, $(z,m) \in s(v,p,0) \Rightarrow -pz \geq -pz' \ \forall z'$ and $c(z)=0$

(B) Suppose $c(z) \in [0,\infty]$. In this case, the money commodity can be used as an input, but not as an output.

1. The definition of the subdifferential $\partial v(z)$ becomes:

$$\partial[-c](z) = \{p : -pz - c(z) \geq -pz' - c(z') \ \ \forall z', \, c(z) < \infty\}$$

   Therefore, $p \in \partial v(z) \Rightarrow -pz - c(z) \geq -pz' - c(z') \, \forall z'$ and $c(z) < \infty$. (z is in the production set)

2. The corresponding definition of the profit function is:

$$\begin{aligned} \pi^*(p) &= \sup_{m,z} \{m - c(z) : pz + m = 0\} \\ &= \sup_{z} \{-pz - c(z) : c(z) < \infty\} \end{aligned}$$

   Therefore, $\pi^*(p) = -pz - c(z) \Rightarrow -pz - c(z) \geq -pz' - c(z') \, \forall z'$ and $c(z) < \infty$

3. The corresponding definition of the supply correspondence is

$$\begin{aligned} s(c, p, 0) &= \arg\max \{m - c(z) : m + pz = 0, \, c(z) < \infty\} \\ &= \arg\max \{-pz - c(z) : c(z) < \infty\} \end{aligned}$$

   Therefore, $(z, m) \in s(v, p, 0) \Rightarrow -pz - c(z) \geq -pz' - c(z') \, \forall z'$ and $c(z) < \infty$

2 Roy's Identity and Hotelling's Lemma

2.1 Part (a)

What assumptions on v are necessary to demonstrate that v^* is convex? Demonstrate. Suggestion: From the definition of v^*, it follows that for every z and p,

$$v(z) - v^*(p) \leq pz.$$

2.1.1 Answer to (a)

There need be no restrictions on v in order to ensure that v^* is convex.
Proof. Let z_0 maximize under p_0, let z_1 maximize under p_1, and let z^* maximize under $p_\lambda = (1-\lambda) p_0 + \lambda p_1$.

$$\begin{aligned} v^*(p_0) &= v(z_0) - p_0 z_0 \geq v(z^*) - p_0 z^* \\ &\Rightarrow (1-\lambda) v^*(p_0) \geq (1-\lambda)[v(z^*) - p_0 z^*] \\ v^*(p_1) &= v(z_1) - p_1 z_1 \geq v(z^*) - p_1 z^* \\ &\Rightarrow \lambda v^*(p_1) \geq \lambda [v(z^*) - p_1 z^*] \end{aligned}$$

Therefore,

$$\begin{aligned} (1-\lambda) v^*(p_0) + \lambda v^*(p_1) &\geq (1-\lambda)[v(z^*) - p_0 z^*] + \lambda [v(z^*) - p_1 z^*] \\ &= (1-\lambda) v(z^*) + \lambda v(z^*) - [(1-\lambda) p_0 + \lambda p_1] z^* \\ &= v(z^*) - p_\lambda z^* \\ &= v^*(p_\lambda) \end{aligned}$$

That is, v^* is convex.

Alternatively, we have:

$$\begin{aligned}
v^*\left((1-\lambda)p_0+\lambda p_1\right) &= \sup_z\left\{v(z)-\left[(1-\lambda)p_0+\lambda p_1\right]z\right\} \\
&\leq \sup_z\left\{(1-\lambda)v(z)-(1-\lambda)p_0 z\right\}+\sup_z\left\{\lambda v(z)-\lambda p_1 z\right\} \\
&= (1-\lambda)\sup_z\left\{v(z)-p_0 z\right\}+\lambda\sup_z\left\{v(z)-p_1 z\right\} \\
&= (1-\lambda)v^*(p_0)+\lambda v^*(p_1)
\end{aligned}$$

That is, v^* is convex regardless of what properties v has. ■

2.2 Part $(b.1)$

Define $z \in \partial\left[-v^*\right](p)$. [The definition is similar to $\partial v(z)$, except that the roles of prices and quantities are reversed].

2.2.1 Answer to $(b.1)$

Following the suggestion, we have that

$$\partial\left[-v^*\right](p)=\left\{z \mid -v^*(p)-pz \geq -v^*(p')-p'z \quad \forall p'\right\}$$

2.3 Part $(b.2)$

Use the definition in $(b.1)$ to show that $p \in \partial v(z)$ implies $z \in \partial\left[-z^*\right](p)$.

2.3.1 Answer to $(b.2)$

Let p' be arbitrary. Then we have that

$$v^*(p') \geq v(z)-p'z \quad \forall z \tag{1}$$

Now, assume $p \in \partial v(z)$. Then, by question 1 part (a),

$$\begin{aligned}
v^*(p) &= v(z)-pz \quad \text{or} \\
v(z) &= v^*(p)+pz
\end{aligned} \tag{2}$$

Substituting (1) into (2), we have:

$$\begin{aligned}
v^*(p') &\geq v^*(p)+pz-p'z \text{ or} \\
-v^*(p)-pz &\geq -v^*(p')-p'z
\end{aligned}$$

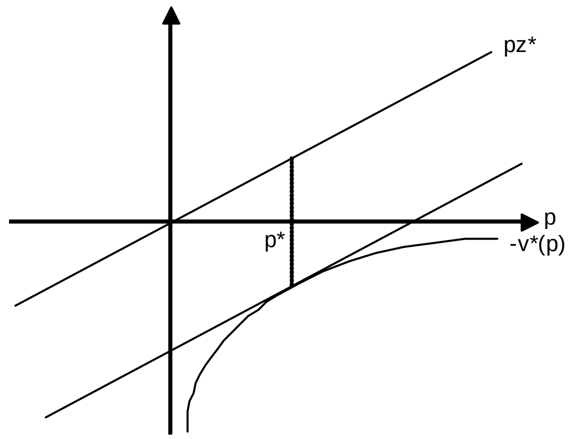
Since p' was arbitrary, this holds $\forall p'$.
Therefore, $z \in \partial\left[-v^*\right](p)$.

2.4 Part (c)

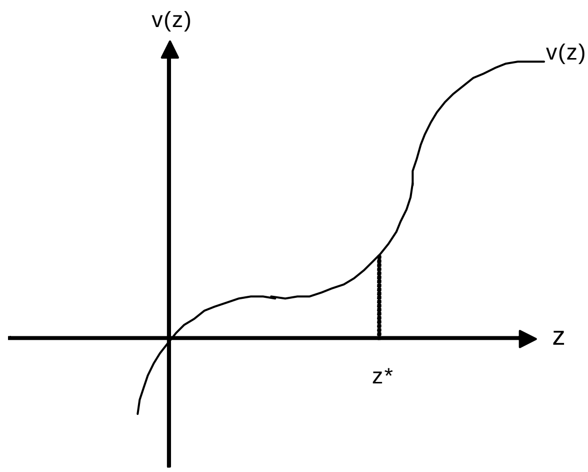
Show by diagram or example that $z \in \partial\left[-v^*\right](p)$ need not imply $p \in \partial v(z)$. What condition on v would make that implication valid?

2.4.1 Answer to (c)

The problem we run into here is that $v^*(p)$ is convex (and hence $-v^*(p)$ concave) no matter what properties $v(z)$ has. Therefore, $\partial[-v^*](p)$ is non-empty for all $p > 0$. Consider the following situation where $z^* \in \partial[-v^*](p^*)$:



It may be the case that $v(z)$ looks like:



In which case $\partial v(z^*) = \emptyset$. Therefore, it is possible that $z \in \partial[-v^*](p)$ but $p \notin \partial v(z)$.

This issue disappears when we impose the restriction that $v(z)$ be a concave function, a property which allows the subdifferential $\partial v(z)$ to be nonempty for all z.

2.5 Part (d)

Interpreting v as representing the non-money part of the utility function for quasilinear utility, explain how the result in $(b.2)$ is related to Roy's Identity. In what sense is the result in $(b.2)$ more general than Roy's Identity and in what sense is it less general?

2.5.1 Answer to (d)

Roy's identity states that if $v(p, I) = \max_x \{U(x) : p \cdot x \leq I\}$, then:

$$\frac{\frac{-\partial v}{\partial p}}{\frac{\partial v}{\partial I}} = x^*(p, I)$$

Adapting this to quasilinear preferences and the notation that we have developed in class, we have $v^*(p,m) = \max_z \{v(z) + m : -p \cdot z \le m\}$ and since $\frac{\partial v^*}{\partial m} = 1$.

$$\frac{-\frac{\partial v^*}{\partial p_i}}{\frac{\partial v^*}{\partial m}} = -\frac{\partial v^*}{\partial p_i} = z_i \ \forall i$$

That is, it is possible to recover the demand function from the indirect utility function. Analogously, $z \in \partial[-v^*](p)$. This result is more general than Roy's identity since Roy's identity requires differentiability of v^*. The concept of subdifferentials requires no such condition. Roy's identity is more general in the sense that it does not depend on the fact that there are no income effects.

2.6 Part (e)

Interpreting $v = -c$ as representing the producer's problem, explain how the result in $(b.2)$ is related to Hotelling's Lemma. Using case (A) in part (c) of Question 1, in what sense is the result in (b) more general than Hotelling's Lemma? Is there a sense in which it is less general?

2.6.1 Answer to (e)

Hotelling's lemma states that if $\pi(p) = \max_{z \in Z}\{-p \cdot z\}$, then:

$$\frac{\partial \pi}{\partial p_i} = -z_i$$

That is, given the profit function, we can derive the input demand or output supply functions. Analogously, $z \in \partial\pi(p)$. Similar to the above result, this result is more general than Hotelling's lemma in the sense that the concept of the subdifferential does not require differentiability of the profit function. I cannot figure out a sense in which it is less general.

3 An Edgeworth Box Version of Quasi Linearity

There are two individuals and two commodities. The utility functions are

$$\begin{aligned} u_1(x_1, m_1) &= 2(x_1)^{\frac{1}{2}} + m_1 \\ u_2(x_2, m_2) &= (x_2)^{\frac{1}{2}} + m_2 \end{aligned}$$

where $x_i, m_i \ge 0$, $i \in \{1, 2\}$. There are 100 units of each commodity.

3.1 Part (a)

Derive the locus of efficient (Pareto optimal) allocations and give a precise illustration of their shape and position in an Edgeworth Box. From the locus of efficient allocations, give a precise illustration of the utility possibility frontier.

3.1.1 Answer to (a)

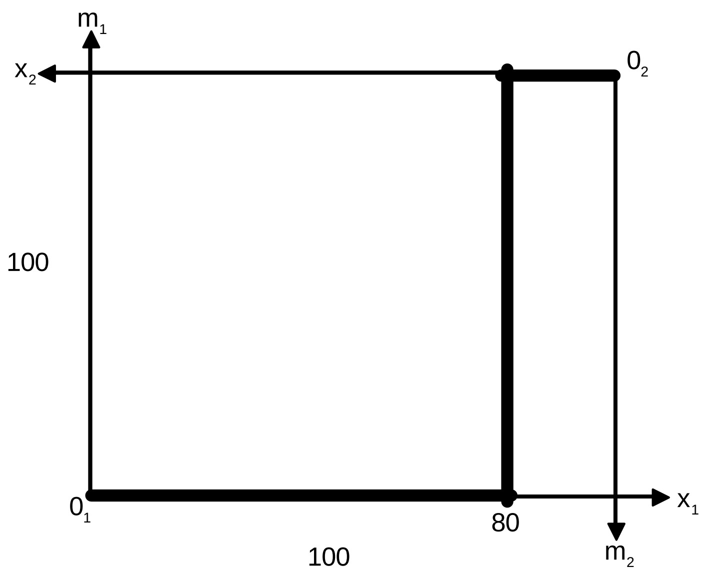
In the interior, $MRS^1 = MRS^2$

$$\begin{aligned} MRS^1 &= MRS^2 \\ \frac{\partial u_1/\partial x_1}{\partial u_1/\partial m_1} &= \frac{\partial u_2/\partial x_2}{\partial u_2/\partial m_2} \\ \frac{2\frac{1}{2}(x_1)^{-\frac{1}{2}}}{1} &= \frac{\frac{1}{2}(x_2)^{-\frac{1}{2}}}{1} \\ 2x_2^{\frac{1}{2}} &= x_1^{\frac{1}{2}} \\ x_1 &= 4x_2 \end{aligned}$$

Since these allocations are in the Edgeworth Box, we have that

$$\begin{aligned} x_1 + x_2 &= 100 \\ 4x_2 + x_2 &= 100 \\ x_2 &= 20 \\ x_1 &= 80 \end{aligned}$$

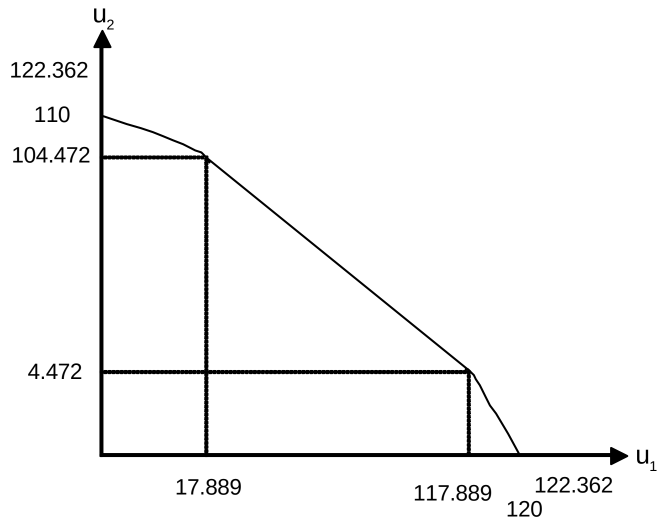
This holds $\forall m_1 \in [0, 100]$, so we have that the locus of efficient allocations looks like:



The utility possibility frontier is derived by evaluating the utilities for consumer 1 and consumer 2 at the Pareto optimal points.

$$\begin{aligned} u_1(0,0) &= 0,\ u_2(100,100) = \sqrt{100} + 100 = 110 \\ u_1(x_1,0) &= 2\sqrt{x_1},\ u_2(100-x_1,100) = 100 + \sqrt{100-x_1} \quad 0 < x_1 < 80 \\ u_1(80,0) &= 2\sqrt{80} \approx 17.889,\ u_2(20,100) = 100 + \sqrt{20} \approx 104.472 \\ u_1(80,m_1) &= 2\sqrt{80} + m_1,\ u_2(20,100-m_1) = 100 - m_1 + \sqrt{20} \quad x_1 = 80 \\ &\Rightarrow u_1(80,m_1) + u_2(20,100-m_1) = 2\sqrt{80} + m_1 + 100 - m_1 + \sqrt{20} \\ &= 100 + 10\sqrt{5} \\ u_1(80,100) &= 2\sqrt{80} + 100,\ u_2(20,0) = \sqrt{20} \\ u_1(x_1,100) &= 100 + 2\sqrt{x_1},\ u_2(100-x_1,0) = \sqrt{100-x_1} \quad 80 < x_1 < 100 \\ u_1(100,100) &= 100 + 2\sqrt{100} = 120,\ u_2(0,0) = 0 \end{aligned}$$

Graphically, this looks like:



3.2 Part (*b*)

Let the initial endowments of the x and m commodities for individual 1 be 25 and 75, respectively, which means the endowments for 2 are 75 and 25. Find the price-taking equilibrium quantities, prices, and utilities.

3.2.1 Answer to (*b*)

If we assume the Pareto optimal allocation corresponding to this endowment is in the interior, we have:

$$\frac{p_x}{1} = MRS^1|_{x_1=80} = MRS^2|_{x_2=20} = \frac{1}{\sqrt{80}}$$

At these prices, it follows that:

$$\begin{aligned} p_x x_1 + m_1 &= p_x \omega_x^1 + \omega_m^1 \\ \frac{80}{\sqrt{80}} + m_1 &= \frac{25}{\sqrt{80}} + 75 \\ m_1 &= \frac{-55}{\sqrt{80}} + 75 \\ &\approx 68.85 \end{aligned}$$

Therefore,

$$m_2 = 100 - m_1 = 100 - 68.85 \approx 31.15$$

The equilibrium prices, quantities, and utilites are therefore:

$$\begin{aligned} p_x &= \frac{1}{\sqrt{80}} \\ p_m &= 1 \\ (x_1, m_1) &= (80, 68.85) \\ (x_2, m_2) &= (20, 31.15) \\ u_1 &= 68.85 + 2\sqrt{80} \approx 86.74 \\ u_2 &= 31.15 + \sqrt{20} \approx 35.62 \end{aligned}$$

As a quick verification, we see that:

$$u_1 + u_2 = 86.74 + 35.62 \approx 122.36$$

That is, this lies on the utility frontier.

3.3 Part (*c*)

Modify the Edgeworth Box above by assuming that m_i could be ANY positive or negative quantity and that the initial endowment of the money commodity is 0 for each individual. Redraw the Edgeworth "box," find the locus of efficient allocations, and illustrate the utility possibility frontier. What change is there in the description of price-taking equilibrium?

3.3.1 Answer to (*c*)

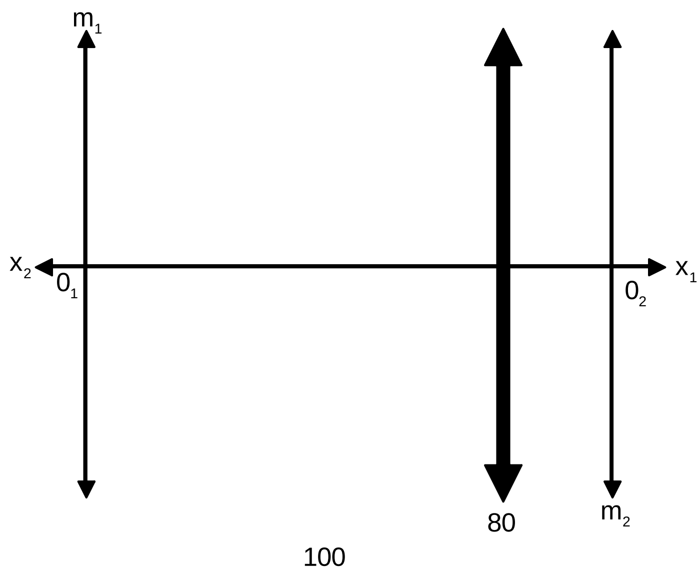
If we allow m_1 and m_2 to take on any positive or negative value, the efficient allocations will still be of the form

$$\begin{aligned} MRS^1 &= MRS^2 \\ \frac{\partial u_1/\partial x_1}{\partial u_1/\partial m_1} &= \frac{\partial u_2/\partial x_2}{\partial u_2/\partial m_2} \\ \frac{2\frac{1}{2}(x_1)^{-\frac{1}{2}}}{1} &= \frac{\frac{1}{2}(x_2)^{-\frac{1}{2}}}{1} \\ 2x_2^{\frac{1}{2}} &= x_1^{\frac{1}{2}} \\ x_1 &= 4x_2 \end{aligned}$$

And, since $\omega_x = 100$,

$$\begin{aligned} x_1 + x_2 &= 100 \\ 4x_2 + x_2 &= 100 \\ x_2 &= 20 \\ x_1 &= 80 \end{aligned}$$

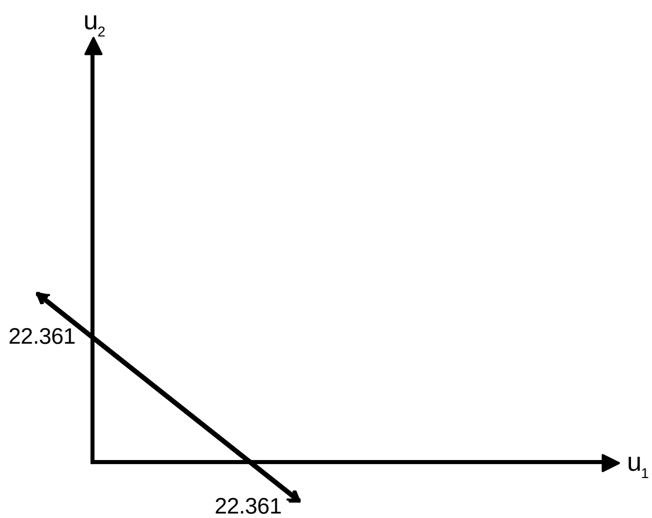
Therefore, the Edgeworth "Box" will look like:



The utility possibility frontier can be derived by calculating u_1 and u_2 at all the Pareto efficient allocations.

$$\begin{aligned} u_1(80, m_1) &= m_1 + 2\sqrt{80},\ u_2(20, -m_1) = -m_1 + \sqrt{20} \quad m_1 \in \mathbb{R} \\ u_1(80, m_1) + u_2(20, -m_1) &= m_1 + 2\sqrt{80} - m_1 + \sqrt{20} = 10\sqrt{5} \approx 22.361 \end{aligned}$$

Graphically, this looks like:



Obviously, not all such points can be supported by prices, since if there is no trade,

$$\begin{aligned} u_1\left(\omega_x^1, 0\right) &= 2\sqrt{\omega_x^1} \geq 0 \\ u_2\left(\omega_x^2, 0\right) &= \sqrt{\omega_x^2} \geq 0 \end{aligned}$$

Thus, any Pareto optimal allocation in which one of the consumers receives negative utility cannot be supported by any allocation with $\omega_x^i \geq 0$ for any set of prices, as it would be rational not to trade.

4 Computing PTE and Marginal Products

The set of individuals is $I = B \cup S$ and there is only one non-money commodity. All buyers are the same and have tastes given by

$$v_b(z_b) = \begin{cases} \beta z_b - \frac{1}{2}\gamma z_b^2 & \text{if } z_b \geq 0, \\ -\infty & \text{otherwise} \end{cases}$$

all sellers are the same and have tastes given by $v_s = -c_s$ where

$$c_s(z_s) = \begin{cases} \frac{1}{2}\sigma z_s^2 & \text{if } z_s \leq 0 \\ \infty & \text{otherwise} \end{cases}$$

Let $|B| = M$ and $|S| = N$. Therefore, the model is completely described by the five parameters $(\beta, \gamma, \sigma, M, N)$. The first three are strictly positive numbers.

Answers to the following questions are to be given without filling in the values of the parameters, but to get started you might want to fill in values for β, γ, σ, say $(1, 1, 1)$.

4.1 Part (a)

Find the PTE price p of the non-money commodity as a function of the parameters and also the values of z_b and z_s. (Why is the price unique? Why are z_b and z_s unique?)

4.1.1 Answer to (a)

As with any general equilibrium problem, we must find aggregate demand, aggregate supply, and equate them to find equilibrium prices and quantities. Built into the utility functions in this problem is that $\omega_z^s = \omega_z^b = 0$.

First, consider the buyer's problem:

$$\max_{z_b} U(z_b, m_b) \text{ s.t. } p_z z_b + m_b = 0$$

$$\begin{aligned} & \max_{z_b} v_b(z_b) - p_z z_b \\ = \;& \max_{z_b} -p_z z_b + \begin{cases} \beta z_b - \frac{1}{2}\gamma z_b^2 & z_b \geq 0 \\ -\infty & \text{otherwise} \end{cases} \\ = \;& \max_{z_b} -p_z z_b + \beta z_b - \frac{1}{2}\gamma z_b^2 \text{ s.t. } z_b \geq 0 \end{aligned}$$

Taking first order conditions, we have:

$$\begin{aligned} (z_b) \quad : \quad & -p_z + \beta - \gamma z_b = 0 \\ \Rightarrow \quad & \gamma z_b = \beta - p_z \\ \Rightarrow \quad & z_b = \frac{\beta - p_z}{\gamma} \end{aligned}$$

Therefore, each consumer has a demand function $z_b = \frac{\beta - p_z}{\gamma}$. Total demand is therefore $M z_b = M \frac{\beta - p_z}{\gamma}$.

Next, consider the seller's problem:

$$\begin{aligned} & \max_{z_s} -p_z z_s - c(z_s) \\ = \;& \max_{z_s} -p_z z_s - \begin{cases} \frac{1}{2}\sigma z_s^2 & z_s \leq 0 \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

$$\max_{z_s} -p_z z_s - \frac{1}{2}\sigma z_s^2 \text{ s.t. } z_s \leq 0$$

Taking first order conditions, we have:

$$\begin{aligned}(z_s) \quad : \quad & -p_z - \sigma z_s = 0 \\ \Rightarrow \quad & z_s = -\frac{p_z}{\sigma}\end{aligned}$$

Individual supply is $z_s = -\frac{p_z}{\sigma}$. Total supply is therefore $Nz_s = -N\frac{p_z}{\sigma}$

The market clearing conditions are

$$\begin{aligned} Mz_b + Nz_s &= 0 \\ Mz_b &= -Nz_s \end{aligned}$$

Therefore, we have:

$$\begin{aligned} M\frac{\beta - p_z}{\gamma} &= N\frac{p_z}{\sigma} \\ \frac{M\beta}{\gamma} &= \frac{Np_z}{\sigma} + \frac{Mp_z}{\gamma} \\ &= p_z\left(\frac{N}{\sigma} + \frac{M}{\gamma}\right) \\ &= p_z\left(\frac{N\gamma + M\sigma}{\sigma\gamma}\right) \\ p_z &= \left(\frac{\sigma\gamma}{N\gamma + M\sigma}\right)\frac{M\beta}{\gamma} \\ &= \frac{\sigma\beta M}{N\gamma + M\sigma} \end{aligned}$$

At this price, we have:

$$\begin{aligned} z_b &= \frac{\beta - p_z}{\gamma} = \frac{\beta}{\gamma} - \frac{\sigma\beta M}{N\gamma^2 + M\sigma\gamma} \\ &= \frac{\beta\gamma N + \beta\sigma M - \sigma\beta M}{N\gamma + M\sigma\gamma} \\ &= \frac{\beta N}{\gamma N + \sigma M} \\ z_s &= -\frac{p_z}{\sigma} = -\frac{\sigma\beta M}{\sigma\left(N\gamma + M\sigma\right)} \\ &= -\frac{\beta M}{\gamma N + \sigma M} \end{aligned}$$

The price is unique by convexity of preferences and convexity of the production set. z_b and z_s are unique due to convexity and since everyone is identical.

4.2 Part (b)

Without using the results in (a), find the maximum gains from trade, i.e.,

$$v_I(0) = \max\left\{Mv_b(z_b) + Nv_s(z_s) : Mz_b + Nz_s = 0\right\}.$$

Note: This expression for the maximum presumes that it can be achieved by treating all buyers identically and all sellers identically. What is the justification for that presumption?

4.2.1 Answer to (b)

This part is simply a constrained maximization problem

$$\begin{aligned}
v_I(0) &= \max_{z_b,z_s}\{Mv_b(z_b)+Nv_s(z_s) : Mz_b+Nz_s=0\} \\
&= \max_{z_s}\left\{Mv_b\left(-\frac{N}{M}z_s\right)+Nv_s(z_s)\right\} \\
&= \max_{z_s}\left\{M\beta\left[-\frac{N}{M}z_s\right]-\frac{M\gamma}{2}\left[-\frac{N}{M}z_s\right]^2-\frac{N}{2}\sigma z_s^2\right\} \\
&= \max_{z_s}\left\{-\beta N z_s-\frac{\gamma N^2}{2M}z_s^2-\frac{N}{2}\sigma z_s^2\right\}
\end{aligned}$$

Taking first order conditions, we have:

$$\begin{aligned}
(z_s) \quad : \quad & -\beta N-\frac{\gamma N^2}{M}z_s-N\sigma z_s=0 \\
\Rightarrow \quad & \beta+\frac{\gamma N}{M}z_s+\sigma z_s=0 \\
\Rightarrow \quad & z_s\left(\sigma+\frac{\gamma N}{M}\right)=-\beta \\
\Rightarrow \quad & z_s\left(\frac{\sigma M+\gamma N}{M}\right)=-\beta \\
\Rightarrow \quad & z_s=\frac{-\beta M}{\sigma M+\gamma N}
\end{aligned}$$

A quick glance verifies that as long as $\beta,\gamma\geq 0$, $z_s\leq 0$ as required.
Evaluating $v_I(0)$ at $z_s=\frac{-\beta M}{\sigma M+\gamma N}$, we have:

$$\begin{aligned}
v_I(0) &= -\beta N z_s-\frac{1}{2}\gamma\frac{N^2}{M}z_s^2-\frac{1}{2}N\sigma z_s^2 \\
&= -\beta N\frac{-\beta M}{\sigma M+\gamma N}-\frac{\gamma N^2}{2M}\left(\frac{-\beta M}{\sigma M+\gamma N}\right)^2-\frac{N\sigma}{2}\left(\frac{-\beta M}{\sigma M+\gamma N}\right)^2 \\
&= \left(\frac{-\beta M}{\sigma M+\gamma N}\right)^2\left[\left(-\beta N\frac{\sigma M+\gamma N}{-\beta M}\right)-\frac{\gamma N^2}{2M}-\frac{N\sigma}{2}\right] \\
&= \left(\frac{-\beta M}{\sigma M+\gamma N}\right)^2\left[\frac{2N(\sigma M+\gamma N)}{2M}-\frac{\gamma N^2}{2M}-\frac{MN\sigma}{2M}\right] \\
&= \left(\frac{-\beta M}{\sigma M+\gamma N}\right)^2\left[\frac{2\sigma NM+2\gamma N^2-\gamma N^2-\sigma MN}{2M}\right] \\
&= \left(\frac{-\beta M}{\sigma M+\gamma N}\right)^2\left[\frac{N(\sigma M+\gamma N)}{2M}\right] \\
&= \frac{\beta^2 MN}{2\sigma M+2\gamma N}
\end{aligned}$$

In summary,

$$v_I(0)=\frac{\beta^2 MN}{2\sigma M+2\gamma N}$$

4.3 Part (c)

Show that the allocations of the non-money commodity in (a) and (b) are the same. How does the set of all Pareto-optimal allocations compare to the allocation in (a)?

4.3.1 Answer to (c)

From (a), we have:

$$\begin{aligned} z_b &= \frac{\beta N}{\gamma N + \sigma M} \\ z_s &= -\frac{\beta M}{\gamma N + \sigma M} \end{aligned}$$

Evaluating the sum of utility $M\left[v_b\left(z_b\right) - p_z z_b\right] + N\left[-p_z z_s - c_s\left(z_s\right)\right]$ at these values gives us:

$$\begin{aligned} & M\left[v_b\left(z_b\right) - p_z z_b\right] + N\left[-p_z z_s - c_s\left(z_s\right)\right] \\ = \ & M\left[\frac{\beta^2 N}{\gamma N + \sigma M} - \frac{\gamma}{2}\left(\frac{\beta N}{\gamma N + \sigma M}\right)^2 - p_z \frac{\beta N}{\gamma N + \sigma M}\right] \\ & + N\left[p_z \frac{\beta M}{\gamma N + \sigma M} - \frac{\sigma}{2}\left(\frac{-\beta M}{\gamma N + \sigma M}\right)^2\right] \\ = \ & M\left(\frac{\beta N}{\gamma N + \sigma M}\right)^2\left[\beta\left(\frac{\gamma N + \sigma M}{\beta N}\right) - \frac{\gamma}{2}\right] - \frac{N\sigma}{2}\left(\frac{-\beta M}{\gamma N + \sigma M}\right)^2 \\ = \ & \frac{M\beta^2 N^2}{\left(\gamma N + \sigma M\right)^2}\left[\frac{2\gamma N + 2\sigma M - \gamma N}{2N}\right] - \frac{N\sigma\beta^2 M^2}{2\left(\gamma N + \sigma M\right)^2} \\ = \ & \frac{M\beta^2 N(\gamma N + 2\sigma M)}{2\left(\gamma N + \sigma M\right)^2} - \frac{N\sigma\beta^2 M^2}{2\left(\gamma N + \sigma M\right)^2} = \frac{\beta^2\gamma M N^2 + 2\beta^2\sigma M^2 N - \beta^2\sigma M^2 N}{2\left(\gamma N + \sigma M\right)^2} \\ = \ & \frac{\beta^2\gamma M N^2 + \beta^2\sigma M^2 N}{2\left(\gamma N + \sigma M\right)^2} = \frac{\beta^2 M N\left(\gamma N + \sigma M\right)}{2\left(\gamma N + \sigma M\right)^2} \\ = \ & \frac{\beta^2 M N}{2\gamma N + 2\sigma M} \end{aligned}$$

That is, the allocation from (a) is Pareto-optimal since it maximizes the sum of utility.
From (b), we have:

$$\begin{aligned} z_s &= -\frac{\beta M}{\gamma N + \sigma M} \\ z_b &= \left(-\frac{N}{M}\right)\left(-\frac{\beta M}{\gamma N + \sigma M}\right) \\ &= \frac{\beta N}{\gamma N + \sigma M} \end{aligned}$$

Which corresponds to the same allocation as in (a).

4.4 Part (d)

Assume $M = N = k$ and denote the economy by v^k. Show that for each k, the price-taking equilibrium p^k for v^k has $p^k = p^1$.

4.4.1 Answer to (d)

From (a), we have that, when everyone is optimizing, the market clearing price for the non-money commodity as a function of M and N is

$$p_z = \frac{\sigma\beta M}{N\gamma + M\sigma}$$

If we let $M = N = k$, then the market clearing price is:

$$\begin{aligned} p_z &= \frac{\sigma\beta k}{\gamma k + \sigma k} = \frac{k}{k}\frac{\sigma\beta}{\gamma + \sigma} \\ &= \frac{\sigma\beta}{\gamma + \sigma} \end{aligned}$$

Since this is not a function of k, I conclude that for each k, the price-taking equilibrium p^k for v^k has $p^k = p^1$ as desired.

4.5 Part (e)

Show that

$$MP_i\left(v^k\right) > v_i^*\left(p\right),$$

for each $i \in \{b, s\}$.

4.5.1 Answer to (e)

Given that the optimal allocations are (when $M = N = k$):

$$\begin{aligned} v_b &= \frac{\beta}{\gamma + \sigma} \\ v_s &= -\frac{\beta}{\gamma + \sigma} \end{aligned}$$

we have that:

$$\begin{aligned} v_b^*\left(p_z\right) &= \beta z_b - \frac{1}{2}\gamma z_b^2 - p_z z_b \\ &= \beta\frac{\beta}{\gamma + \sigma} - \frac{\gamma}{2}\left(\frac{\beta}{\gamma + \sigma}\right)^2 - \left(\frac{\sigma\beta}{\gamma + \sigma}\right)\left(\frac{\beta}{\gamma + \sigma}\right) \\ &= \frac{2\beta^2\left(\gamma + \sigma\right)}{2\left(\gamma + \sigma\right)^2} - \frac{\gamma\beta^2}{2\left(\gamma + \sigma\right)^2} - \frac{2\sigma\beta^2}{2\left(\gamma + \sigma\right)^2} \\ &= \frac{2\beta^2\left(\gamma + \sigma\right) - \gamma\beta^2 - 2\sigma\beta^2}{2\left(\gamma + \sigma\right)^2} \\ &= \frac{\beta^2\gamma}{2\left(\gamma + \sigma\right)^2} \end{aligned}$$

If we define, in the spirit of part (b)

$$v_{M,N}\left(0\right) = \frac{\beta^2 MN}{2\sigma M + 2\gamma N}$$

Then we know that when $M = N = k$,

$$\begin{aligned}
MP_b &= v_{k,k}(0) - v_{k-1,k}(0) \\
&= \frac{\beta^2 k^2}{2\sigma k + 2\gamma k} - \frac{\beta^2 (k-1) k}{2\sigma (k-1) + 2\gamma k} \\
&= \frac{\beta^2 k^2}{2\sigma k + 2\gamma k} - \frac{\beta^2 k^2 - \beta^2 k}{2\sigma k + 2\gamma k - 2\sigma} \\
&= \frac{\beta^2 k^2 (2\sigma k + 2\gamma k - 2\sigma)}{(2\sigma k + 2\gamma k)(2\sigma k + 2\gamma k - 2\sigma)} - \frac{\left(\beta^2 k^2 - \beta^2 k\right)(2\sigma k + 2\gamma k)}{(2\sigma k + 2\gamma k - 2\sigma)(2\sigma k + 2\gamma k)} \\
&= \frac{2\beta^2\sigma k^3 + 2\beta^2\gamma k^3 - 2\beta^2\sigma k^2 - 2\beta^2\sigma k^3 - 2\beta^2\gamma k^3 + 2\beta^2\sigma k^2 + 2\beta^2\gamma k^2}{(2\sigma k + 2\gamma k)(2\sigma k + 2\gamma k - 2\sigma)} \\
&= \frac{2\beta^2\gamma k^2}{(2\sigma k + 2\gamma k)(2\sigma k + 2\gamma k - 2\sigma)} \\
&= \frac{\beta^2\gamma k}{(\sigma + \gamma)(2\sigma k + 2\gamma k - 2\sigma)}
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
MP_b - v_b^* &= \frac{\beta^2\gamma k}{(\sigma + \gamma)(2\sigma k + 2\gamma k - 2\sigma)} - \frac{\beta^2\gamma}{2(\gamma + \sigma)^2} \\
&= \frac{2\beta^2\gamma k (\sigma + \gamma)}{2(\sigma + \gamma)^2 (2\sigma k + 2\gamma k - 2\sigma)} - \frac{\beta^2\gamma (2\sigma k + 2\gamma k - 2\sigma)}{2(\gamma + \sigma)^2 (2\sigma k + 2\gamma k - 2\sigma)} \\
&= \frac{2\beta^2\gamma\sigma k + 2\beta^2\gamma^2 k - 2\beta^2\gamma\sigma k - 2\beta^2\gamma^2 k + 2\beta^2\gamma\sigma}{2(\sigma + \gamma)^2 (2\sigma k + 2\gamma k - 2\sigma)} \\
&= \frac{\beta^2\gamma\sigma}{2(\sigma + \gamma)^2 (\sigma k + \gamma k - \sigma)} \\
&= \frac{\beta^2\gamma\sigma}{2(\sigma + \gamma)^2 (\sigma (k-1) + \gamma k)}
\end{aligned}$$

Thus, whenever $\beta, \sigma, \gamma > 0$ and $k \geq 1$, we have that $MP_b - v_b^* > 0$

Similarly, for the sellers,

$$\begin{aligned}
v_s^*(p_z) &= -\left(\frac{\sigma\beta}{\gamma + \sigma}\right)\left(-\frac{\beta}{\gamma + \sigma}\right) - \frac{1}{2}\sigma\left(-\frac{\beta}{\gamma + \sigma}\right)^2 \\
&= \frac{\sigma\beta^2}{(\gamma + \sigma)^2} - \frac{\sigma}{2}\frac{\beta^2}{(\gamma + \sigma)^2} \\
&= \frac{\beta^2}{(\gamma + \sigma)^2}\sigma - \frac{\beta^2}{(\gamma + \sigma)^2}\frac{\sigma}{2} \\
&= \left(\frac{\beta}{\gamma + \sigma}\right)^2 \left(\sigma - \frac{\sigma}{2}\right) \\
&= \frac{\sigma}{2}\left(\frac{\beta}{\gamma + \sigma}\right)^2
\end{aligned}$$

And

$$\begin{aligned}
MP_s &= v_{k,k}(0) - v_{k,k-1}(0) \\
&= \frac{\beta^2 k^2}{2\sigma k + 2\gamma k} - \frac{\beta^2 k (k-1)}{2\sigma k + 2\gamma (k-1)} \\
&= \frac{\beta^2 k^2 (2\sigma k + 2\gamma k - 2\gamma) - \left(\beta^2 k^2 - \beta^2 k\right)(2\sigma k + 2\gamma k)}{(2\sigma k + 2\gamma k)(2\sigma k + 2\gamma k - 2\gamma)} \\
&= \frac{2\beta^2\sigma k^3 + 2\beta^2\gamma k^3 - 2\beta^2\gamma k^2 + 2\beta^2\sigma k^2 + 2\beta^2\gamma k^2 - 2\beta^2\sigma k^3 - 2\beta^2\gamma k^3}{(2\sigma k + 2\gamma k)(2\sigma k + 2\gamma k - 2\gamma)} \\
&= \frac{\beta^2 \sigma k}{(\sigma + \gamma)(2\sigma k + 2\gamma k - 2\gamma)}
\end{aligned}$$

Therefore we have that

$$\begin{aligned}
MP_s - v_s^*(p_z) &= \frac{\beta^2 \sigma k}{(\sigma + \gamma)(2\sigma k + 2\gamma k - 2\gamma)} - \frac{\sigma\beta^2}{2(\gamma + \sigma)^2} \\
&= \frac{2\beta^2 \sigma k (\gamma + \sigma)^2 - \sigma\beta^2 (\sigma + \gamma)(2\sigma k + 2\gamma k - 2\gamma)}{2(\gamma + \sigma)^2 (\sigma + \gamma)(2\sigma k + 2\gamma k - 2\gamma)} \\
&= \frac{\beta^2 \sigma \gamma}{2(\gamma + \sigma)^2 (\sigma k + \gamma k - \gamma)} \\
&= \frac{\beta^2 \gamma \sigma}{2(\gamma + \sigma)^2 (\sigma k + \gamma (k-1))}
\end{aligned}$$

Thus, whenever $\beta, \gamma, \sigma > 0$ and $k \geq 1$ we have that $MP_s - v_s^*(p_z) > 0$

4.6 Part (f)

Show that $MP_i\left(v^k\right) \to v_i^*(p)$ as $k \to \infty$ for each $i \in \{b, s\}$.

4.6.1 Answer to (f)

Let $i = b$. Then we have:

$$MP_b - v_b^*(p_z) = \frac{\beta^2 \gamma \sigma}{2(\sigma + \gamma)^2 (\sigma k + \gamma k - \sigma)} \to 0 \text{ as } k \to \infty$$

That is, $MP_i\left(v^k\right) \to v_i^*(p)$ as $k \to \infty$.
Let $i = s$. Then we have:

$$MP_s - v_s^*(p_z) = \frac{\beta^2 \gamma \sigma}{2(\gamma + \sigma)^2 (\sigma k + \gamma k - \gamma)} \to 0 \text{ as } k \to \infty$$

That is, $MP_i\left(v^k\right) \to v_i^*(p)$ as $k \to \infty$.