Proposition 1 *If $p >> 0$ and $y_j^* \in \eta_j(p)$, then y_j^* is efficient with respect to Y_j.*

Proof. In order to get a contradiction, suppose y_j^* is not efficient with respect to Y_j. Then $\exists\, y_j' \in Y_j$ such that $y_j' \geq y_j^* \wedge y_j' \neq y_j^*$. Then $py_j' > py_j^*$, which is a contradiction to $y_j^* \in \eta_j(p) = \{y_j : py_j = \pi(p)\}$. ■

Proposition 2 $\pi(p) = \sum_j \pi_j(p)$

I will give two separate proofs for this proposition. The first one requires a lemma.

Lemma 3 *Let A, B be sets. Define $C = \{a + b : a \in A, b \in B\}$. Then $\sup C = \sup A + \sup B$.*

Proof. By the definition of sup, $\forall a \in A$, $a \leq \sup A$. $\forall b \in B$, $b \leq \sup B$. Let $a \in A$, $b \in B$ be arbitrary. Then $a + b \leq \sup A + b \leq \sup A + \sup B$. That is, $\sup A + \sup B$ is an upper bound for C. Then since $\sup C$ is the least upper bound of C, we have

$$\sup C \leq \sup A + \sup B \tag{1}$$

Fix $b \in B$, and let $a \in A$ be arbitrary. Then since $a + b \in C$, we know

$$\begin{aligned} a + b &\leq \sup C \\ a &\leq \sup C - b \end{aligned}$$

That is, $\sup C + b$ is an upper bound for A. Since $\sup A$ is the least upper bound for A and since b was arbitrarily fixed, we have that $\sup A \leq \sup C - b \ \forall b \in B$. Rearranging, we have

$$b \leq \sup C - \sup A \ \forall b$$

That is, $\sup C - \sup A$ is an upper bound for B. Since $\sup B$ is the least upper bound for B, we have

$$\begin{aligned} \sup B &\leq \sup C - \sup A \\ \sup A + \sup B &\leq \sup C \qquad (2) \end{aligned}$$

Combining (1) and (2), we have $\sup C = \sup A + \sup B$ as desired. ■

Proof. (Proof of proposition) Without loss of generality, assume $N = 2$. Then we have

$$\begin{aligned} \pi(p) &= \sup_{y \in Y} \{p \cdot y\} = \sup_{y} \{p \cdot y : y = y_1 + y_2 \wedge y_1 \in Y_1 \wedge y_2 \in Y_2\} \\ &= \sup_{y_1 \in Y_1} \{p \cdot y_1\} + \sup_{y_2 \in Y_2} \{p \cdot y_2\} \\ &= \pi_1(p) + \pi_2(p) \end{aligned}$$

Where the lemma was u sed in the second to last step. ■

Proof. (Alternate proof of proposition) In order to get a contradiction, suppose $\pi(p) > \sum_j \pi_j(p)$. Then for some $\tilde{y} \in Y = \left\{y : y = \sum_j y_j,\ y_j \in Y_j \ \forall j\right\}$,

$$\begin{aligned} p \cdot \tilde{y} &> \sum_{j=1}^{N} p \cdot y_j \\ \sum_{j=1}^{N} p \cdot \tilde{y}_j &> \sum_{j=1}^{N} p \cdot y_j \end{aligned}$$

Then $\exists j$ such that $p \cdot \tilde{y}_j > p \cdot y_j = \pi_j(p)$, which is a contradiction. Therefore, $\pi(p) = \sum_j \pi_j(p)$. ■

Proposition 4 *If $\left[(x_i^*), \left(y_j^*\right), p\right]$ is a price taking equilibrium for ε, then it is a price taking equilibrium for $\hat{\varepsilon}$.*

Proof. Since $Y \subset \hat{Y}$, it must be the case that $\sup_{y \in Y} \{p \cdot y\} \leq \sup_{y \in \hat{Y}} \{p \cdot y\}$. In order to get a contradiction, suppose $\sup_{y \in \hat{Y}} \{p \cdot y\} > \sup_{y \in Y} \{p \cdot y\}$. Then $\exists\ \hat{y} \in \hat{Y}$ satisfying $p \cdot \hat{y} = \sup_{y \in \hat{Y}} \{p \cdot y\}$. Recall that $\hat{Y} = \left\{y : y = \sum_{k=1}^{\ell+1} \alpha_k y_k \wedge \sum_{k=1}^{\ell+1} \alpha_k = 1 \wedge \alpha_k \geq 0\ \forall k\right\}$. Then,

$$\begin{aligned} \sup_{y \in Y} \{p \cdot y\} &< p \cdot \hat{y} \\ &= p \cdot \left[\sum_{k=1}^{\ell+1} \alpha_k y_k\right] \end{aligned}$$

Where $y_k \in Y\ \forall k$. Then it must be the case that $\exists\ k$ such that

$$p \cdot y_k > \sup_{y \in Y} \{p \cdot y\}$$

Which is a contradiction. Therefore, $\sup_{y \in \hat{Y}} \{p \cdot y\} = \sup_{y \in Y} \{p \cdot y\}$. ■

Proposition 5 *If p is an equilibrium price for ε, it is an equilibrium price for ε^2 if ε is replica invariant.*

Proof. $\sum_i \xi_i (p, w_i(p)) - \sum_j \eta_j(p) = \omega \iff 2\sum_i \xi_i(p, w_i(p)) - 2\sum_j \eta_j(p) = 2\omega$. ■

Proposition 6 *If $\exists$ a price taking equilibrium for ε, then ε is replica invariant.*

Definition 7 *$\hat{v}(z) = \sup\left\{\sum_k \lambda_k v(z_k) : \sum_k \lambda_k z_k = z,\ \lambda_k \geq 0,\ \sum_k \lambda_k = 1\right\}$ is the concavified utility function. That is, it is the smallest concave function lying on or above the graph of v.*

Proposition 8 *If $p \in \partial v(z)$, then $v(z) = \hat{v}(z)$*

Proof. By definition of $\hat{v}$, it follows that $\hat{v}(z) \geq v(z)$. In order to get a contradiction, suppose $\hat{v}(z) > v(z)$. Then $\sum_k \lambda_k v(z_k) > v(z)$. Then z lies in the non-concave portion of $v(z)$. Thus, it cannot be the case that $p \in \partial v(z)$, which is a contradiction. Therefore, $v(z) = \hat{v}(z)$. ■

Proposition 9 *$v(z) - p \cdot z = v^*(p) \iff p \in \partial v(z)$.*

Proof. $(\Rightarrow)$ Suppose $v(z) - p \cdot z = v^*(p)$. Then $v(z) - p \cdot z \geq v(z') - p \cdot z'\ \forall z'$. That is, $p \in \partial v(z)$.

$(\Leftarrow)$ Suppose $p \in \partial v(z)$. Then $v(z) - p \cdot z \geq v(z') - p \cdot z'\ \forall z'$. That is, $v(z) - p \cdot z = v^*(p)$. ■

Proposition 10 *Let $[(z_i), p]$ be a price taking equilibrium such that $\sum_i z_i = 0$ and $v_i^*(p) = v_i(z_i) - p \cdot z_i$ $\forall i$. Then (z_i) is Pareto optimal.*

This proposition requires a lemma.

Lemma 11 *Fix (z_i) such that $\sum_i z_i = 0$. Then $\forall p$, $\sum_i v_i(z_i) \le \sum_i v_i^*(p)$.*

Proof. Fix (z_i) such that $\sum_i z_i = 0$. Let p be arbitrary. By definition, $v_i^*(p) \ge v_i(z_i) - p \cdot z_i \ \forall i \ \forall z_i$. Therefore,

$$\begin{aligned} \sum_i v_i^*(p) &\ge \sum_i [v_i(z_i) - p \cdot z_i] \\ &= \sum_i v_i(z_i) - p \sum_i z_i \\ &= \sum_i v_i(z_i) \end{aligned}$$

Which is the desired result. ■

Proposition 12 *If $\sum_i v_i^*(p) = \sum_i v_i(z_i)$, then $\sum_i v_i(z_i) = v_I(0)$. That is, (z_i) is PO.*

Proof. Follows directly from duality theory since $v_I(0) = \sup_{z_i} \{\sum_i v_i(z_i) : \sum_i z_i = 0\} = \inf_p \{\sum_i v_i^*(p)\}$. ■

Proposition 13 $p \in \partial v(z) \Rightarrow v(z) = \hat{v}(z)$

Proof. In order to get a contradiction, suppose $v(z) < \hat{v}(z)$

$$\begin{aligned} v(z) - p \cdot z &< \sum_{k=1}^{\ell+1} \lambda_k v(z_k) - \sum_{k=1}^{\ell+1} \lambda_k p \cdot z_k \text{ s.t. } \sum_{k=1}^{\ell+1} \lambda_k z_k = z \wedge \sum_{k=1}^{\ell+1} \lambda_k = 1 \wedge \lambda_k \ge 0 \ \forall k \\ \sum_{k=1}^{\ell+1} \lambda_k [v(z) - p \cdot z] &< \sum_{k=1}^{\ell+1} \lambda_k [v(z_k) - p \cdot z_k] \end{aligned}$$

Therefore $\exists\ k$ such that

$$v(z) - p \cdot z < v(z_k) - p \cdot z_k$$

Which is a contradiction. Therefore, $v(z) = \hat{v}(z)$. ■

Proposition 14 $v^*(p) = \hat{v}^*(p)$

Proof. In order to get a contradiction, assume $v^*(p) < \hat{v}^*(p)$.

$$v^*(p) = v(z) - p \cdot z < \hat{v}^*(p) = \sum_{k=1}^{\ell+1} \lambda_k [v(z_k) - p \cdot z_k]$$

$$\sum_{k=1}^{\ell+1} \lambda_k [v(z) - p \cdot z] < \sum_{k=1}^{\ell+1} \lambda_k [v(z_k) - p \cdot z_k]$$

Therefore $\exists\ k$ such that

$$v^*(p) = v(z) - p \cdot z < v(z_k) - p \cdot z_k$$

Which is a contradiction. Therefore, $v^*(p) = \hat{v}^*(p)$. ■

Proposition 15 *Non-infinitessimally,* $-p \cdot z_i \leq v_{-i}(-z_i) - v_{-i}(0) \equiv SOC_{-i}(-z_i)$.

Proof. Suppose $p \in \partial v_{-i}(-z_i)$. Then, by definition,

$$\begin{aligned} v_{-i}(-z_i) - p(-z_i) &\geq v_{-i}(0) - p(0) \\ v_{-i}(-z_i) + p \cdot z_i &\geq v_{-i}(0) \\ -p \cdot z_i &\leq v_{-i}(-z_i) - v_{-i}(0) \end{aligned}$$

Which is the desired inequality. ■

Proposition 16 *Non-infinitessimally,* $v_i^*(p) \leq v_I(0) - v_{-i}(0) \equiv MP_i$.

Proof.

$$\begin{aligned} MP_i &= v_I(0) - v_{-i}(0) \\ &= \sup\{v_i(z_i) + v_{-i}(-z_i)\} - v_{-i}(0) \\ &= \sup\{v_i(z_i) + v_{-i}(-z_i) - v_{-i}(0)\} \\ &= \sup\{v_i(z_i) + \Delta v_{-i}(-z_i)\} \end{aligned}$$

Note that

$$\begin{aligned} v_i^*(p) &= v(z_i) - p \cdot z_i \quad \text{for } z_i \text{ optimal} \\ v(z_i) &= v_i^*(p) - p \cdot (-z_i) \quad \text{for } z_i \text{ optimal} \end{aligned}$$

Therefore, we have:

$$MP_i = v_i^*(p) - p \cdot z_i + \Delta v_{-i}(-z_i) \text{ for } z_i \text{ optimal}$$

This gives us Joonsuk's Theorem:

$$\begin{aligned} MP_i - v_i^*(p) &= \Delta v_{-i}(-z_i) - p \cdot (-z_i) \\ MP_i - v_i^*(p) &= SOC_{-i}(-z_i) - p \cdot (-z_i) \end{aligned}$$

By previous proposition, we know that

$$\begin{aligned} -p \cdot z_i &\leq SOC_{-i}(-z_i) \\ SOC_{-i}(-z_i) - p \cdot (-z_i) &\geq 0 \end{aligned}$$

Therefore, by Joonsuk's Theorem:

$$\begin{aligned} MP_i - v_i^*(p) &\geq 0 \\ MP_i &\geq v_i^*(p) \end{aligned}$$

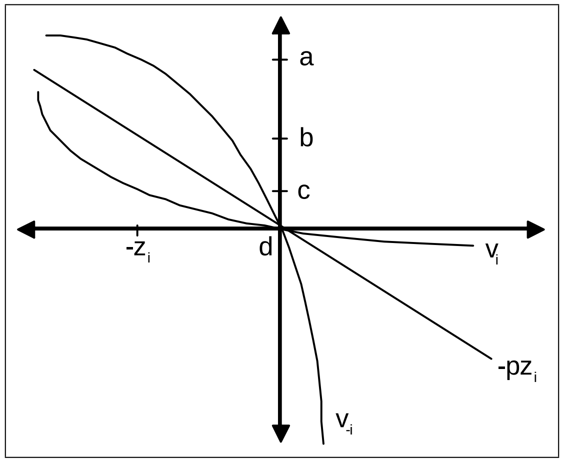
Which is the desired inequality. ■

Remark 17 *Since in a PTE,* $\sum_i v_i(z_i) = \sum_i v_i^*(p) = v_I(0)$, *we have that*

$$\sum_i MP_i \geq \sum_i v_i^*(p) = v_I(0)$$

In general, we cannot have full appropriation, because we will eventually run out of the pie!

Remark 18 *Graphically,*



$$
\begin{aligned}
MP_i &= v_I(0) - v_{-i}(0) = v_i(z_i) + v_{-i}(-z_i) - v_{-i}(0) \\
&= ac \\
v_i^*(p) &= v_i(z_i) - p \cdot z_i = bc \\
SOC_i(-z_i) &= v_{-i}(-z_i) - v_{-i}(0) = ad \\
-p \cdot (-z_i) &= bd
\end{aligned}
$$

To verify Joonsuk's theorem, we have:

$$
\begin{aligned}
MP_i - v_i^*(p) &= ac - bc = ab \\
SOC_{-i}(-z_i) - p \cdot z_i &= ad - bd = ab \\
MP_i - v_i^*(p) &= SOC_{-i}(-z_i) - p \cdot (-z_i)
\end{aligned}
$$