Homework due Friday at 12:00pm in the grad. lab.

Consumer:

prices for non-money commodities. $p_m \equiv 1$

$-\ d(v, p, w)$ $\quad v: \mathbb{R}^{\ell} \to \mathbb{R} \cup \{-\infty\}$

utility

$v(z) = -\infty \leftarrow$ not a feasible trade (negative consumption)

$\hookrightarrow$ we are only interested in $z \ni v(z) > -\infty$

$$d(v,p,w) = \{ (z,m) \mid \underbrace{p \cdot z}_{\text{market value of trades}} + m = w,\ v(z) + m \geq v(z') + m' \ \forall (z', m') \ni pz' + m' = w \}$$

Special case: $d(v,p,0)$ - wealth is zero

$$d(v,p,w) = d(v,p,0) + (0,w)$$

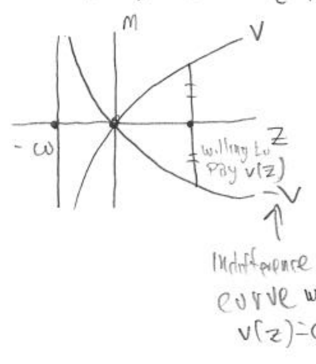
$$(z^*, m^*) = (z', m') + (0, w) \Rightarrow z^* = z' \Rightarrow \text{No income effects for non-money commodities}$$

$\Rightarrow$ WLog, we can assume $w = 0$.

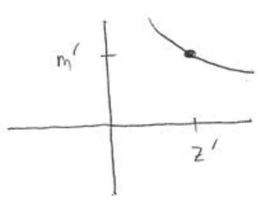
Inverse demand:

$m = -pz$ since $w = 0$ and $pz + m = 0$

$$d^{-1}(v, z, 0) = \{ p \mid (z, -pz) \in d(v, p, 0) \}$$



$(z,m) \succsim (z',m') \iff v(z') + m' \leq v(z) + m$



$d^{-1}(v,z,0)$ can be empty if v is not concave.

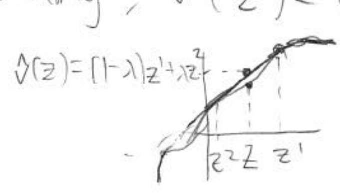
Let $\partial v(z) = \{ p \mid v(z) - pz \geq v(z') - pz' \ \forall z' \} \equiv d^{-1}(v, z, 0)$

$\llcorner$ subdifferential of v at z,

Define conc $v = \hat{v} \ \ni \ \hat{v}(z) = \sup \{ \sum \lambda_k v(z_k) \mid \sum \lambda_k z_k = z,\ \lambda_k \geq 0,\ \sum \lambda_k = 1 \}$

"concavified utility function" - smallest concave function lying on or above v.

Clearly, $\hat{v}(z) \geq v(z)\ \forall z$



Proposition: If $p \in \partial v(z)$, $v(z)=\hat{v}(z)$

If p is in the subdifferential of v at z, then z is not in the non-concave portion of the utility fcn.

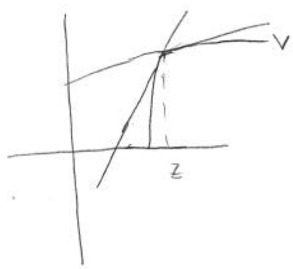
Proof: By contradiction. $v(z) < \hat{v}(z) \Rightarrow p \notin \partial v(z)$. Fill in the details.

Uniqueness of $\partial v(z)$

Suppose v is concave and $\partial v(z) = \{p\}$

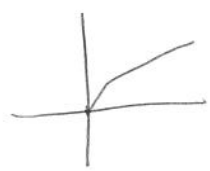
$$\nabla v(z) = \left(\frac{\partial v(z)}{\partial z_1}, \dots, \frac{\partial v(z)}{\partial z_l}\right) = p \Rightarrow \partial v(z) = \nabla v(z)$$

$\{$ if $l=1$, $\nabla v(z) = v'(z) = \frac{dv}{dz}(z)$ $\}$



v is not differentiable at z.

This notion handles boundary problems.



Indirect utility $(v,p,w) \equiv \overline{U}(v,p,w)$
$= v(z) + m \quad (z,m) \in d(v,p,w)$

Indirect utility $(v,p,0) \equiv \overline{U}(v,p,0)$
$= v(z) + m - w$

$\Rightarrow \overline{U}(v,p,w) - \overline{U}(v,p,0) = w$

$\Rightarrow$ Wlog, we can assume $w=0$. this contains the restriction

Define $\overline{U}(v,p,0) = v^*(p) = \sup_z \{v(z) - pz\}$

$v^*(p)$ conjugate of v

$v(z) \quad v: \mathbb{R}^l \to \mathbb{R}$

$v^*(p) \quad v^*: \mathbb{R}^l_+ \to \mathbb{R}$

Proposition: $v(z) - pz = v^*(p) \iff p \in \partial v(z)$

Pf: This is just a restatement of the definitions.

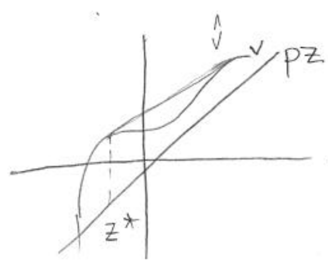
$$\frac{\partial u_i(x)}{\partial x_k} = \lambda p_k$$

$$\frac{\partial v_i(z)}{\partial z_k} = p_k$$. FOCs for QL utility: $p = \nabla v(z)$

$v^*(p)$ is convex
$\Rightarrow v^*(\lambda p + (1-\lambda)p') \le (1-\lambda)v^*(p) + \lambda v^*(p')$ } confirm this!

Relationship b/t $v^*(p)$ and $(\hat{v} = \text{conc } v)^*(p)$?
$v^*(p) = \hat{v}^*(p)$ by proposition: If $p \in \partial v(z)$, then $v(z) = \hat{v}(z)$

$$v^*(p) = \sup_z \{v(z) - pz\}$$



v^* maximize this positive difference

This will still be the z^* of conc v

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application to a PTE:

We cannot separate out E from $\mathcal{E}$

$I = \{1, \ldots, n\}$

v_i completely characterizes i's preferences.

$v_i : \mathbb{R}^\ell \to [-\infty, \infty)$

Cannot disconnect u-fcn from endowments since endowments are built into the u-fcns.

$\mathcal{E} \to \mathbf{v} = (v_1, \ldots, v_n)$

$[(z_i, m_i)] = [(z_1, m_1), \ldots, (z_n, m_n)]$ - allocations

$(p, 1)$ - prices

Such that

Utility is maximized (equivalent ways of writing this)

1) $(z_i, m_i) \in d(v_i, p, 0) \quad \forall i$

or

2) $p \in \partial v_i(z_i) \quad \forall i$

or

3) $v_i(z_i) - p z_i = v_i^*(p)$

The trade is attainable:

$$\sum_{i=1}^{n} (z_i, m_i) = (0, 0) \in \mathbb{R}^{l+1}$$

Is this equilibrium efficient?

An allocation that is feasible and maximizes the sum of utilities is efficient under QL preferences

$\max \{ \sum v_i(z_i) \mid \sum z_i = 0 \}$

$v_i(\lambda z + (1-\lambda) z') > \lambda v(z) + (1-\lambda) v(z'), \quad z \neq z'$

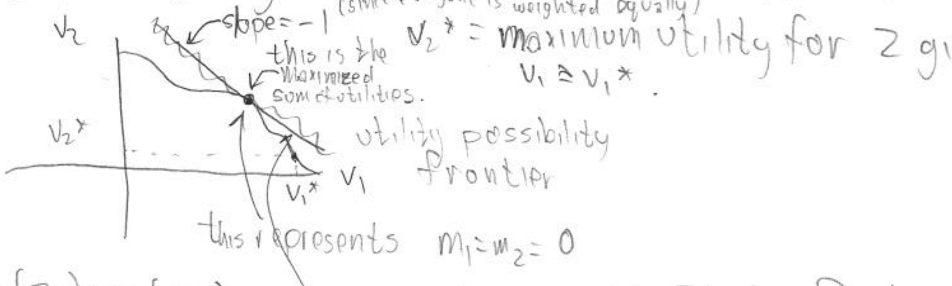
└ strict concavity ⇒ uniqueness

1st step | Maximize the sum of utility

2nd step | Transfer utility in the form of $(m_1, \dots, m_n) \ni \sum_i m_i = 0$

Let $n = 2$

v_2^* = maximum utility for z given that $v_1 \geq v_1^*$.



$$\max_{z_1} \; v_1(z_1) + v_2(-z_1)$$