Econ142: Probabilistic Microeconomics

Problem Set 6

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1 Question 1

For each of the following rules, check whether it is dictatorial. If not, then it must violate at least one of the Arrow's assumptions. Find which. In all cases, there are n individuals.

Recall that a social decision rule $\succ$ follows Arrow's assumptions if it satisfies:

1. Unanimity. If $\forall i$, $a \succ_i b$, then $a \succ b$.
2. Transitivity. If $a \succ b$ and $b \succ c$, then $a \succ c$.
3. Independence of Irrelevant Alternatives. $\succ$ over a and b depend only on individual preferences over a and b.

1.1 Part (*a*)

$a \succ b$ whenever at least $n-1$ members of society strictly prefer a to b. Otherwise, $a \sim b$. (Assume $n \geq 4$)

1.1.1 Answer

This clearly satisfies unanimity since if all n members of society strictly prefer a to b, then at least $n-1$ members strictly prefer a to b and thus $a \succ b$.

This also satisfies the independence of irrelevant alternatives condition since the only information taken into account is the individual preferences over a and b.

Therefore, it must be the case that this social decision rule does not obey transitivity. To see this, consider the case when $n = 4$ and the social options are a, b, c, and d. Suppose further that preferences are the following:

$$a \succ_1 b \succ_1 c \succ_1 d$$

$$b \succ_2 c \succ_2 d \succ_2 a$$

$$c \succ_3 d \succ_3 a \succ_3 b$$

$$d \succ_4 a \succ_4 b \succ_4 c$$

Then we have that $a \succ b$ (since $a \succ_i b \ \forall i \in \{1,3,4\}$), $b \succ c$ (since $b \succ_i c \ \forall i \in \{1,2,4\}$), $c \succ d$ (since $c \succ_i d \ \forall i \in \{1,2,3\}$) and $d \succ a$ (since $d \succ_i a \ \forall i \in \{2,3,4\}$). Thus, $a \succ b \succ c \succ d \succ a$, which violates transitivity.

1.2 Part (*b*)

$a \succ b$ whenever all members of society strictly prefer a to b. Otherwise, $a \sim b$.

1.2.1 Answer

Obviously, this social decision rule satisfies unanimity. It also depends only on individual preferences over a given set of outcomes, so it satisfies the independence of irrelevant alternatives axiom. In addition, it is not dictatorial. To see this, suppose there are only two individuals (1 and 2), two social policies (a and b), and preferences are as follows:

$$a \succ_1 b$$

$$b \succ_2 a$$

Then $a \sim b$, which does not coincide with either person's preferences, which establishes the claim. By deduction, it must be that this social decision rule violates transitivity.

Suppose there are two individuals (1 and 2), three social policies ($a, b,$ and c), and preferences are as follows:

$$a \succ_1 b \succ_1 c$$

$$c \succ_2 a \succ_2 b$$

Then we have:

$$a \succ b \sim c \sim a$$

Which violates transitivity (it cannot be the case that $a \succ a$).

1.3 Part (c)

There are two alternatives, a and b. A coin is flipped. If H, $a \succ b$, and if T, $b \succ a$.

1.3.1 Answer

The flip of a coin is not related to individual preferences over the two alternatives, yet it is being used to make the social decision. Therefore, this violates the independence of irrelevant alternatives condition.

1.4 Part (d)

Suppose the set of options is $a_1, ..., a_k$. Then

- $a_1 \succeq a_2$ if and only if $a_1 \succeq_1 a_2$; and
- For all other pairs of options, $a_i \succeq a_j$ if and only if $a_i \succeq_2 a_j$.

That is, the social preferences are the same as those of person 2, except for the case when a_1 and a_2 are compared, in which case social preferences are the same as those of person 1.

1.4.1 Answer

First, it is important to note that these preferences also satisfy $a_2 \succ a_1$ if and only if $a_2 \succ_1 a_1$ and for all other pairs of options, $a_i \succ a_j$ if and only if $a_i \succ_2 a_j$.

To see this, suppose $a_2 \succ_1 a_1$. This is the same as saying $a_1 \not\succeq_1 a_2$ which occurs if and only if $a_1 \not\succeq a_2$, which is the same as saying $a_2 \succ a_1$. A similar argument establishes the analogous property for person 2.

Now, suppose preferences are as follows:

$$a_2 \succ_1 a_1$$

$$a_1 \succ_2 a_3 \succ_2 a_2$$

Then we have

$$a_1 \succ a_3 \succ a_2 \succ a_1$$

This obviously violates transitivity (since it is impossible for $a_1 \succ a_1$).

2 Question 2

Society is composed of two individuals, with utility functions u_1 and u_2. The utility possibilities curve (UPC) is given by

$$u_1 + 2u_2 = 300$$

The social welfare function W is given by

$$W(u_1, u_2) = u_1 u_2$$

What is the best allocation of utility from a social point of view?

2.0.2 Answer

The problem of the social planner with preferences $W(u_1, u_2) = u_1 u_2$ is to

$$\max u_1 u_2$$

$$s.t.\ u_1 + 2u_2 \leq 300$$

We can set up a Lagrangian to solve this:

$$L(u_1, u_2, \lambda) = u_1 u_2 + \lambda (300 - u_1 - 2u_2)$$

The first order conditions are thus:

$$\begin{aligned} (u_1) &: \quad u_2 - \lambda = 0 \\ (u_2) &: \quad u_1 - 2\lambda = 0 \\ (\lambda) &: \quad 300 - u_1 - 2u_2 = 0 \end{aligned}$$

This gives us

$$\begin{aligned} u_2 &= \lambda \\ \frac{u_1}{2} &= \lambda \text{ or} \\ 2u_2 &= u_1 \end{aligned}$$

Which, when plugged into the constraint gives

$$\begin{aligned} 4u_2 &= 300 \\ u_2 &= \frac{300}{4} = 75 \\ u_1 &= 2u_2 = 2(75) = 150 \end{aligned}$$

3 Question 3

Do the same for the case

$$W(u_1, u_2) = u_1 + u_2$$

3.0.3 Answer

Since both the objective and the constraint are linear, the solution will (most likely) be on the boundary. Therefore, we must be careful when it comes to checking boundary conditions. The Lagrangian is:

$$L(u_1, u_2, \lambda) = u_1 + u_2 + \lambda(300 - u_1 - 2u_2)$$

Which gives first order conditions (and complementary slackness conditions):

$$\begin{aligned}
(u_1) &: \quad 1 - \lambda \le 0 \text{ and } u_1 \ge 0 \text{ with } (1-\lambda)u_1 = 0 \\
(u_2) &: \quad 1 - 2\lambda \le 0 \text{ and } u_2 \ge 0 \text{ with } (1-2\lambda)u_2 = 0 \\
(\lambda) &: \quad 300 - u_1 - 2u_2 \ge 0 \text{ and } \lambda \ge 0 \text{ with } (300 - u_1 - 2u_2)\lambda = 0
\end{aligned}$$

Graphically, we can see that the optimum will occur when $u_1 > 0$ and $u_2 = 0$ and the constraint is binding. That is, $\lambda > 0$.

This will give us:

$$\begin{aligned}
300 - u_1 - 2u_2 &= 300 - u_1 - 2(0) = 0 \\
u_1 &= 300 \\
u_2 &= 0
\end{aligned}$$

Intuitively, this makes sense since if the social planner values a unit of utility from both individuals equally and it is easier to provide utility to individual 1, then it will simply provide all the utility to individual 1.

4 Question 4

Answer questions 3 and 4 again for the case where the UPC is given by

$$2u_1^2 + u_2^2 = 200$$

If you cannot find exact numbers, make sure at least to write the right equations.

4.0.4 Answer

For question 3, the Lagrangian is:

$$L(u_1, u_2, \lambda) = u_1 u_2 + \lambda\left(200 - 2u_1^2 - u_2^2\right)$$

Which gives us the first order conditions

$$\begin{aligned}
(u_1) &: \quad u_2 - 4u_1\lambda = 0 \\
(u_2) &: \quad u_1 - 2u_2\lambda = 0 \\
(\lambda) &: \quad 200 - 2u_1^2 - u_2^2 = 0
\end{aligned}$$

Simplifying, we have:

$$\begin{aligned}
u_2 &= 4u_1\lambda \\
\frac{u_2}{4u_1} &= \lambda \\
u_1 &= 2u_2\lambda \\
\frac{u_1}{2u_2} &= \lambda
\end{aligned}$$

Which gives us:

$$\begin{aligned}
\frac{u_2}{4u_1} &= \frac{u_1}{2u_2} \\
u_2^2 &= 2u_1^2
\end{aligned}$$

Plugging into the constraint, we have:

$$\begin{aligned} 200 - 2u_1^2 - u_2^2 &= 0 \\ 200 - 4u_1^2 &= 0 \\ u_1^2 &= 50 \\ u_1 &= 5\sqrt{2} \end{aligned}$$

Also, we know that $u_2^2 = 2u_1^2$, which gives us

$$\begin{aligned} u_2^2 &= 2\left(50\right) \\ &= 100 \\ u_2 &= 10 \end{aligned}$$

Therefore, the optimal solution to the modified question 3 is

$$u_1 = 5\sqrt{2} \text{ and } u_2 = 10$$

The Lagrangian for question 4 is

$$L\left(u_1, u_2, \lambda\right) = u_1 + u_2 + \lambda\left(200 - 2u_1^2 - u_2^2\right)$$

Taking first order conditions, we have:

$$\begin{aligned} (u_1) &: \quad 1 - 4u_1\lambda = 0 \\ (u_2) &: \quad 1 - 2u_2\lambda = 0 \\ (\lambda) &: \quad 200 - 2u_1^2 - u_2^2 = 0 \end{aligned}$$

Simplifying we have:

$$\begin{aligned} 1 &= 4u_1\lambda \\ \frac{1}{4u_1} &= \lambda \\ 1 &= 2u_2\lambda \\ \frac{1}{2u_2} &= \lambda \end{aligned}$$

Which gives us:

$$\begin{aligned} \frac{1}{4u_1} &= \frac{1}{2u_2} \\ u_2 &= 2u_1 \end{aligned}$$

Plugging into the constraint, we have:

$$\begin{aligned} 200 - 2u_1^2 - u_2^2 &= 0 \\ 200 - 6u_1^2 &= 0 \\ u_1^2 &= \frac{200}{6} = \frac{100}{3} \\ u_1 &= \frac{10}{\sqrt{3}} \end{aligned}$$

Also, we know that $u_2 = 2u_1$, which gives us:

$$u_2 = \frac{20}{\sqrt{3}}$$

Therefore, the optimal solution is thus

$$u_1 = \frac{10}{\sqrt{3}} \text{ and } u_2 = \frac{20}{\sqrt{3}}$$