Econ142: Probabilistic Microeconomics

Problem Set 3

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The lessons presented in these problems are important and somewhat interesting. The first question shows us that in a world of risk neutral agents, there are no gains from insurance. In particular, this suggests the intuitive result that insurance would only arise in situations where people are risk averse.

The second question shows that in situations where there are people of different degrees of risk aversion, there is a market for risk. Similarly, the third question shows that when people have different beliefs about the risk of a certain assets, there are opportunities for gains from trade.

1 Question 1

Three individuals face (each) the risk $(a, p; b, 1-p)$. These risks are statistically independent. They decide to share their outcomes equally

1.1 Part (a)

What lottery will each one of them face?

1.1.1 Answer

Since these lotteries are independent, it is as if society is facing the following situation:

	p^3	$p^2(1-p)$	$p^2(1-p)$	$p(1-p)^2$	$p^2(1-p)$	$p(1-p)^2$	$p(1-p)^2$	$(1-p)^3$
I	a	a	a	a	b	b	b	b
II	a	a	b	b	a	a	b	b
III	a	b	a	b	a	b	a	b
$Total$	$3a$	$2a+b$	$2a+b$	$a+2b$	$2a+b$	$a+2b$	$a+2b$	$3b$

In lottery form, we have that society faces

$$\left(3a, p^3; 2a+b, 3p^2(1-p); a+2b, 3p(1-p)^2; 3b, (1-p)^3\right)$$

If it is split equally, each receives

$$\left(a, p^3; \frac{2a+b}{3}, 3p^2(1-p); \frac{a+2b}{3}, 3p(1-p)^2; b, (1-p)^3\right)$$

1.2 Part (b)

Suppose they are all risk neutral (that is, each has a utility function of the form $ax+b$). Will this arrangement make them better off?

1.2.1 Answer

Intuitively, a risk neutral person cares only about the expected value of a lottery. We would expect that by simply redistributing the risk, we would not change the expected value of the lottery. Therefore, each individual would be indifferent between insuring and not insuring.

A more precise, mathematical exposition follows: Without loss of generality, we can assume that $u(0) = 0$ for each person.

Without sharing, we have

$$u[(a, p; b, 1-p)] = ap + b(1-p) = ap + b - bp$$

With sharing,

$$\begin{aligned}
& u\left[\left(a, p^3; \frac{2a+b}{3}, 3p^2(1-p); \frac{a+2b}{3}, 3p(1-p)^2; b, (1-p)^3\right)\right] \\
= \;& p^3 a + p^2(1-p)(2a+b) + p(1-p)^2(a+2b) + (1-p)^3 b \\
= \;& p^3 a + \left(p^2 - p^3\right)(2a+b) + p\left(1 - 2p + p^2\right)(a+2b) + \left(1 - 2p + p^2\right)(1-p)b \\
= \;& p^3 a + 2ap^2 + bp^2 - 2ap^3 - p^3 b + \left(p - 2p^2 + p^3\right)(a+2b) + \left(1 - 2p + p^2 - p + 2p^2 - p^3\right)b \\
= \;& p^3 a + 2ap^2 + bp^2 - 2ap^3 - p^3 b + ap - 2ap^2 + ap^3 + 2bp - 4bp^2 + 2bp^3 + b - 3bp + 3bp^2 - bp^3 \\
= \;& ap + b - bp
\end{aligned}$$

Therefore, no one is better off under this arrangement

2 Question 2

John's utility function is $u(x) = \sqrt{x}$, while Ann's utility function is $v(x) = \sqrt[3]{x}$. Ann is facing the lottery $\left(64, \frac{1}{2}; 0, \frac{1}{2}\right)$.

2.1 Part (a)

What is the minimal price Ann is willing to accept for selling this lottery?

2.1.1 Answer

The definition of a certainty equivalent suggests that for any sure amount of money more than the certainty equivalent, a given individual would be better off with the money than with the lottery. For any sure amount of money less than the certainty equivalent, a given individual would be better off with the lottery than with the money. Therefore, for any amount of money $p \geq CE_A[X]$, Ann would be willing to sell this lottery

$$\begin{aligned}
Ev[X] &= \frac{1}{2}(64)^{1/3} = \frac{1}{2}4 = 2 \\
Ev[(8,1)] &= (8)^{1/3} = 2
\end{aligned}$$

Thus, $CE_A[X] = 8$. That is, Ann would be willing to accept any $p \geq 8$ for this lottery.

2.2 Part (b)

What is the maximal price John is willing to pay for this lottery?

2.2.1 Answer

For any amount of money $p \leq CE_J[X]$, John would be willing to purchase this lottery

$$\begin{aligned} Eu[X] &= \frac{1}{2}(64)^{1/2} = \frac{1}{2}8 = 4 \\ Eu[(16,1)] &= (16)^{1/2} = 4 \end{aligned}$$

Thus, $CE_J[X] = 16$. That is, John would be willing to pay any $p \leq 16$ for this lottery.

2.3 Part (c)

Is there a price at which they'll trade?

2.3.1 Answer

They would be willing to trade at any $8 \leq p \leq 16$.

2.4 Part (d)

Will they both be better off?

2.4.1 Answer

For any $8 < p < 16$, they would both be better off. At $p = 8$, John would be better off (and Ann indifferent). At $p = 16$, Ann would be better off (and John indifferent).

3 Question 3

John's utility function is $u(x) = \sqrt{x}$, and this time Ann's utility function is the same, that is $v(x) = \sqrt{x}$. Ann is facing the risk $(64, S; 0, \neg S)$, where S is the event "it will snow tomorrow" and $\neg S$ is the event "it will not snow tomorrow." Ann believes that the probability of S is $\frac{1}{4}$, while John believes that the probability of S is $\frac{1}{2}$.

3.1 Part (a)

What lottery does Ann face (in terms of probabilities)?

3.1.1 Answer

Ann believes she is facing $X_{1/4} = \left(64, \frac{1}{4}; 0, \frac{3}{4}\right)$

3.2 Part (b)

What lottery does John believe Ann is facing?

3.2.1 Answer

John believes Ann is facing $X_{1/2} = \left(64, \frac{1}{2}; 0, \frac{1}{2}\right)$

3.3 Part (c)

What is the minimal price Ann is willing to accept for selling this lottery?

3.3.1 Answer

Ann is willing to accept any $p \geq CE_A\left[X_{1/4}\right]$ for this lottery

$$\begin{aligned} Ev\left[X_{1/4}\right] &= \frac{1}{4}(64)^{1/2} = \frac{1}{4}8 = 2 \\ Ev\left[(4,1)\right] &= (4)^{1/2} = 2 \end{aligned}$$

Thus, Ann would be willing to accept any $p \geq 4$ for this lottery

3.4 Part (d)

What is the maximal price John is willing to pay for this lottery?

3.4.1 Answer

John is willing to pay any $p \leq CE_J\left[X_{1/2}\right]$ for this lottery

$$\begin{aligned} Eu\left[X_{1/2}\right] &= \frac{1}{2}(64)^{1/2} = \frac{1}{2}8 = 4 \\ Eu\left[(16,1)\right] &= (16)^{1/2} = 4 \end{aligned}$$

Thus, John would be willing to pay any $p \leq 16$ for this lottery

3.5 Part (e)

Is there a price at which they'll trade?

3.5.1 Answer

They would be willing to trade at any $4 \leq p \leq 16$.

3.6 Part (f)

Will they both be better off?

3.6.1 Answer

For any $4 < p < 16$, they are both better off. For $p = 4$, John is better off (and Ann is indifferent). For $p = 16$, Ann is better off (and John is indifferent).