Econ142: Probabilistic Microeconomics

Problem Set 2
Question 4

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In calculus classes, we learned that a concave function is a function with a negative second derivative. However, there is a more general definition that is useful for economics. It is especially useful since it does not require that a function be differentiable. If you do not plan on getting a PhD in economics, you may find it less painful to skip to the third property I have listed for concave functions below and then on to the solution of the problem.

In order to understand this definition concavity, it is necessary to define the concept of a convex combination. Intuitively, a convex combination of two points is a "weighted average" of the two points. Graphically, it is the set of points on the line segment containing the two points. Formally, the definition is as follows:

Definition: Let x_0 and x_1 be two points on the real line. A point x_λ is a *convex combination* of x_0 and x_1 if for some $\lambda \in [0,1]$, $x_\lambda = (1-\lambda)\, x_0 + \lambda x_1$.

It may be useful to notice that if we let $\lambda = 0$, then $x_\lambda = (1-0)\, x_0 + 0x_1 = x_0$, and if we let $\lambda = 1$, then $x_\lambda = (1-1)\, x_0 + 1x_1 = x_1$. That is, if we put "all the weight" on x_0, then we get back the point x_0. Similarly, if we put "all the weight" on x_1, then we get back the point x_1. If we put any other amount of weight on the two points, we will get something in between - an average of sorts.

With this concept out of the way, it is now possible to introduce the concept of a concave function. The formal definition of a concave function follows:

Definition: A function f is *concave* if for any two points x_0 and x_1, for all $\lambda \in [0,1]$, we have that $f(x_\lambda) \geq (1-\lambda)\, f(x_0) + \lambda f(x_1)$.

This is equivalent to saying:

1. A concave function lies below any of its tangent lines
2. The line segment connecting any two points on the curve lies beneath the curve
3. The slope of the tangent lines are decreasing (this is the property I will make use of in this problem)

1 Question 4

The decision maker's utility function is concave. Show that

$$\left(576, \frac{1}{2}; 100, \frac{1}{2}\right) \succ \left(676, \frac{1}{2}; 0, \frac{1}{2}\right)$$

Try doing it for the utility function $u(x) = \sqrt{x}$

1.0.1 Answer:

Intuitively, this question is obvious. This question asks us to compare two lotteries with the same expected value, but with different variances. A risk averse individual (someone with a concave utility function) would strictly prefer the lottery with the smaller variance. (We proved a similar result in class involving risk premiums and their relationship to variance)

The approach I will take to this problem is what is called the "magic bullet" approach in math. Towards the beginning of the "proof," you will probably ask yourself "Where did he come up with that?" The problem with this approach is that it is probably somewhat difficult to follow, and rightfully so. In actuality, I worked out this problem in reverse, seeing what conditions on u I would need in order for $\left(576, \frac{1}{2}; 100, \frac{1}{2}\right) \succ \left(676, \frac{1}{2}; 0, \frac{1}{2}\right)$. Nevertheless, here goes:

If we define a function $v(x) = u(x) - u(x - 576)$, we have:

$$v'(x) = u'(x) - u'(x - 576) < 0 \text{ by concavity of } u.$$

It may not be obvious why this follows from concavity. If we look at property 3 of concave functions listed above, we see that the slope of the tangent lines are decreasing. This says that the slope of u at $x - 576$ would be greater than the slope of u at x. Mathematically, this is the same as saying $u'(x - 576) > u'(x)$. This would give us $u'(x) - u'(x - 576) < 0$, which is the desired result.

From this, we know that the function $v(x) = u(x) - u(x - 576)$ is decreasing (since it has a negative derivative). Since v is decreasing, $v(576) > v(676)$. That is

$$\begin{aligned}
v(576) &> v(676) \\
u(576) - u(0) &> u(676) - u(100) \\
u(576) + u(100) &> u(676) + u(0) \quad \text{Multiplying both sides by } \frac{1}{2}: \\
\frac{1}{2}u(576) + \frac{1}{2}u(100) &> \frac{1}{2}u(676) + \frac{1}{2}u(0)
\end{aligned}$$

If we recognize that $Eu\left(576, \frac{1}{2}; 100, \frac{1}{2}\right) = \frac{1}{2}u(576) + \frac{1}{2}u(100)$ and $Eu\left(676, \frac{1}{2}; 0, \frac{1}{2}\right) = \frac{1}{2}u(676) + \frac{1}{2}u(0)$, then we have that

$$\begin{aligned}
Eu\left(576, \frac{1}{2}; 100, \frac{1}{2}\right) &> Eu\left(676, \frac{1}{2}; 0, \frac{1}{2}\right), \text{ which is the same as saying} \\
\left(576, \frac{1}{2}; 100, \frac{1}{2}\right) &\succ \left(676, \frac{1}{2}; 0, \frac{1}{2}\right)
\end{aligned}$$

This is the desired result.

If we restrict attention to the $u = \sqrt{x}$ case, then we have:

$$\begin{aligned}
Eu\left(576, \frac{1}{2}; 100, \frac{1}{2}\right) &= \frac{1}{2}\sqrt{576} + \frac{1}{2}\sqrt{100} \\
&= \frac{1}{2}24 + \frac{1}{2}10 \\
&= 12 + 5 \\
&= 17
\end{aligned}$$

Similarly,

$$\begin{aligned}
Eu\left(676, \frac{1}{2}; 0, \frac{1}{2}\right) &= \frac{1}{2}\sqrt{676} + \frac{1}{2}\sqrt{0} \\
&= \frac{1}{2}26 + \frac{1}{2}0 \\
&= 13
\end{aligned}$$

Therefore, since $17 > 13$, we have that $Eu\left(576, \frac{1}{2}; 100, \frac{1}{2}\right) > Eu\left(676, \frac{1}{2}; 0, \frac{1}{2}\right)$, which is equivalent to saying $\left(576, \frac{1}{2}; 100, \frac{1}{2}\right) \succ \left(676, \frac{1}{2}; 0, \frac{1}{2}\right)$.