Econ142: Probabilistic Microeconomics

Problem Set 2
Question 1

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In class, Professor Segal mentioned a theorem which says that the expected utility functions we can use to represent a given person's preferences over outcomes are "unique up to positive linear transformations." First, note that, in general, a utility function is just a function which represents the order of preference among bundles of goods or lotteries. This theorem implies that if we take a preference ordering and represent it with a utility function u, we can also represent these same preferences with any utility function v of the form $v = \alpha u + \beta$, $\alpha > 0$.

Since we have this flexibility in assigning functions to preference orderings, we can "without loss of generality," impose some structure on the functions we use. (This is sometimes called "normalizing") This question is designed to show us how to use this feature of expected utility functions.

1 Question 1

We have seen that if $\alpha > 0$, then taking the expected utility of u or $\alpha u + \beta$ will provide the same preferences over lotteries. By taking such transformations, show that we can assume, without loss of generality, that:

1.1 Part (a)

$u(0) = 0$

1.1.1 Answer

If we take a set of preferences and assign to it a function u which represents these preferences, then $u(0)$ must equal *something*. I will call this number k. That is, $u(0) = k$ where k is a real number, but we don't know if it is positive, negative, or zero.

By the theorem, we know that we can also represent these preferences by any function of the form $v = \alpha u + \beta$, $\alpha > 0$. Let us take such a function:

$$v(x) = \alpha u(x) + \beta$$

What properties do we want v to satisfy? In this question, all we need is that v satisfies $v(0) = 0$. Therefore, we have one equation in two unknowns:

$$\begin{aligned} v(0) &= \alpha u(0) + \beta \\ &= \alpha k + \beta \\ &= 0 \end{aligned}$$

Solving for α and β, we have:

$$\beta = -\alpha k$$

Since we have more unknowns than equations, this means that there are an infinite number of solutions. (As long as $\alpha > 0$, any β satisfying $\beta = -\alpha k$ works) One example of such a solution is when we set $\alpha = 1$. Then,

$$\begin{aligned} \beta &= -k \\ &= -u(0) \end{aligned}$$

This will give us a function of the form (by plugging in α and β):

$$\begin{aligned} v(x) &= \alpha u(x) + \beta \\ &= u(x) - u(0) \end{aligned}$$

To verify that this function satisfies the criterion we were looking for, let us check that $v(0) = 0$.

$$\begin{aligned} v(0) &= u(0) - u(0) \\ &= 0 \end{aligned}$$

By the theorem, this utility function represents the same preferences as u, so we can "without loss of generality," assume that $u(0) = 0$ by making such a transformation. (It does not matter if we call the utility function u or v, so long as it represents the same preferences)

1.2 Part (b)

$u(0) = 0$ and $u(1) = 1$.

1.2.1 Answer

The remaining two parts of this question are treated the same as the first part, except that there are no longer an infinite number of solutions. If we have a set of preferences, and we represent them with the utility function u, then $u(0)$ must equal *something*. Let us call this number k. That is, $u(0) = k$, where k is a real number. In addition, $u(1)$ must equal *something*. Let us call this number k'. That is, $u(1) = k'$.

By the theorem, we know that we can also represent these preferences by any function of the form $v = \alpha u + \beta$, $\alpha > 0$. Let us take such a function:

$$v(x) = \alpha u(x) + \beta$$

What properties do we want v to satisfy? In this question, we want $v(0) = 0$ and $v(1) = 1$. Therefore, we have two equations in two unknowns:

$$\begin{aligned} v(0) &= \alpha u(0) + \beta \\ &= \alpha k + \beta = 0 \text{ and} && (1) \\ v(1) &= \alpha u(1) + \beta \\ &= \alpha k' + \beta = 1 && (2) \end{aligned}$$

Solving for β in (1) and plugging it into (2), we have:

$$
\begin{aligned}
\beta &= -\alpha k \quad \text{and therefore} \\
1 &= \alpha k' + \beta \\
&= \alpha k' - \alpha k \\
&= \alpha \left(k' - k\right). \quad \text{Solving for } \alpha \text{, we have:} \\
\alpha &= \frac{1}{k' - k}. \quad \text{Plugging this into } \beta \text{, we have:} \\
\beta &= -\alpha k \\
&= -\frac{k}{k' - k}
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
v\left(x\right) &= \alpha u\left(x\right) + \beta \\
&= \frac{1}{k' - k} u\left(x\right) - \frac{k}{k' - k} \\
&= \frac{u\left(x\right) - k}{k' - k}
\end{aligned}
$$

Plugging back in $k = u\left(0\right)$ and $k' = u\left(1\right)$, we have:

$$
v\left(x\right) = \frac{u\left(x\right) - u\left(0\right)}{u\left(1\right) - u\left(0\right)}
$$

To verify that this utility function satisfies the properties we want it to, simply check to see if $v\left(0\right) = 0$ and $v\left(1\right) = 1$.

$$
\begin{aligned}
v\left(0\right) &= \frac{u\left(0\right) - u\left(0\right)}{u\left(1\right) - u\left(0\right)} = \frac{0}{u\left(1\right) - u\left(0\right)} = 0 \\
v\left(1\right) &= \frac{u\left(1\right) - u\left(0\right)}{u\left(1\right) - u\left(0\right)} = 1
\end{aligned}
$$

Therefore, by the theorem, this utility function represents the same preferences as u, so we can "without loss of generality," assume that $u\left(0\right) = 0$ and $u\left(1\right) = 1$ by making such a transformation.

1.3 Part (c)

$u\left(0\right) = 0$ and $u'\left(0\right) = 1$

1.3.1 Answer:

If we have a set of preferences, and we represent them with the utility function u, then $u\left(0\right)$ must equal *something*. Let us call this number k. That is, $u\left(0\right) = k$, where k is a real number. In addition, $u'\left(0\right)$ must equal *something*. Let us call this number k''. That is, $u'\left(0\right) = k''$.

By the theorem, we know that we can also represent these prferences by any function of the form $v = \alpha u + \beta$, $\alpha > 0$. Let us take such a function:

$$
v\left(x\right) = \alpha u\left(x\right) + \beta
$$

What properties do we want v to satisfy? In this question, we want that $v\left(0\right) = 0$ and $v'\left(0\right) = 1$. This gives us two equations in two unknowns:

$$
\begin{aligned}
v(0) &= \alpha u(0) + \beta \\
&= \alpha k + \beta = 0 \quad \text{and} \qquad (3) \\
v'(0) &= \alpha u'(0) \\
&= \alpha k'' = 1 \qquad (4)
\end{aligned}
$$

Using equation (4) to solve for α and plugging it into equation (3), we have:

$$
\begin{aligned}
\alpha k'' &= 1 \text{ which implies} \\
\alpha &= \frac{1}{k''} \quad \text{and therefore} \\
0 &= \alpha k + \beta \\
&= \frac{k}{k''} + \beta \quad \text{which gives us} \\
\beta &= -\frac{k}{k''}
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
v(x) &= \alpha u(x) + \beta \\
&= \frac{1}{k''} u(x) - \frac{k}{k''} \\
&= \frac{u(x) - k}{k''}
\end{aligned}
$$

Plugging back in $k = u(0)$ and $k'' = u'(0)$, we have that

$$
v(x) = \frac{u(x) - u(0)}{u'(0)}
$$

To verify that this utility function satisfies the properties we want it to, simply check to see if $v(0) = 0$ and $v'(0) = 1$.

$$
\begin{aligned}
v(0) &= \frac{u(0) - u(0)}{u'(0)} = \frac{0}{u'(0)} = 0 \\
v'(0) &= \frac{u'(0)}{u'(0)} = 1 \text{ since } v'(x) = \alpha u'(x) = \frac{u'(x)}{u'(0)}
\end{aligned}
$$

Therefore, by the theorem, this utility function represents the same preferences as u, so we can "without loss of generality," assume that $u(0) = 0$ and $u'(0) = 1$ by making such a transformation.