Econ142: Probabilistic Microeconomics

Concave vs. Quasiconcave

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Modern economics is largely based on the mathematical field of convex analysis, so terms like convex, concave, quasiconvex, and quasiconcave tend to come up often. Chances are, the notion of quasiconcavity did not creep up in Econ. 11 or Econ. 101. (It certainly did not in my classes!)

It is rather difficult to explain the distinction between concavity and quasiconcavity without using a lot of math symbols, so I will explain the difference first in terms of mathematical symbols. The definitions I have provided are suitable for any dimension, but it is easiest to think of them on a two-dimensional graph. Afterwards, I will try to give some intuition regarding the difference between the two and perhaps some properties of each.

1 Convex Combinations and Convex Sets

In order to understand the definitions of concavity and quasiconcavity, it is necessary to define the concept of a convex combination. Intuitively, a convex combination of two points is a "weighted average" of the two points. Graphically, it is a point on the line segment containing the two points. Formally, the definition is as follows:

Definition 1 *Let x_0 and x_1 be two points. A point x_λ is a convex combination of x_0 and x_1 if for some $\lambda \in [0,1]$, $x_\lambda = (1-\lambda)\, x_0 + \lambda x_1$.*

It may be useful to notice that if we let $\lambda = 0$, then $x_\lambda = (1-0)\, x_0 + 0x_1 = x_0$, and if we let $\lambda = 1$, then $x_\lambda = (1-1)\, x_0 + 1x_1 = x_1$. That is, if we put "all the weight" on x_0, then we get back the point x_0. Similarly, if we put "all the weight" on x_1, then we get back the point x_1. If we put any other amount of weight on the two points, we will get something in between - an average of sorts.

The properties of concave and quasiconcave functions draw upon the notion of a convex set. Intuitively a convex set is a set that would not change if you were to put a rubber band around it. Mathematically, the definition is:

Definition 2 *A set X is convex if for all $x_0, x_1 \in X$, and for all $\lambda \in [0,1]$, $x_\lambda \in X$ where $x_\lambda = (1-\lambda)\, x_0 + \lambda x_1$.*

At first glance, the mathematical definition may be a little vague. In words, it says that a set is convex if for any two points you take from the set, all the points on the line segment connecting those two points are also contained in the set. In two dimensions, it is easy to show examples of convex and non-convex sets.

Figure 1 shows a set which is convex. Take any two points in the set and draw a line which connects them. You will see that this line is completely contained within the set.

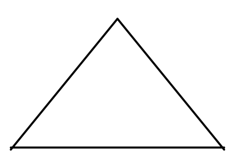

Figure 1: Convex set in two dimensions.

Figure 2, on the other hand, shows a set which is not convex. If you take the points a and b in the set and draw the line segment which connects them, you will see that the line segment is not completely contained within the set. In particular, the point c which lies on that line segment is not in the set.

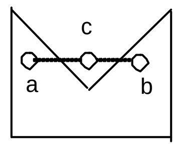

Figure 2: Non-convex set in two dimensions.

2 Concave Functions

Before defining concave functions, so as to avoid confusion, it should be noted that the word "convex" can be used in two (almost completely) different ways. The first is with respect to sets. The second is with respect to functions. The notions of convex sets and convex functions are distinct. You will see that there are some connections between the two if you pursue a higher degree in economics, but for now, think of them as two distinct mathematical notions.

The formal definition of a concave function is as follows:

Definition 3 *A function f is concave if for any two points x_0 and x_1, for all $\lambda \in [0,1]$, we have that $f(x_\lambda) \geq (1-\lambda) f(x_0) + \lambda f(x_1)$.*

Graphically, this says that if you take two points in the domain of the function and draw a line connecting them, the line will lie completely below the function (except at the endpoints). This is shown in Figure 3 below.

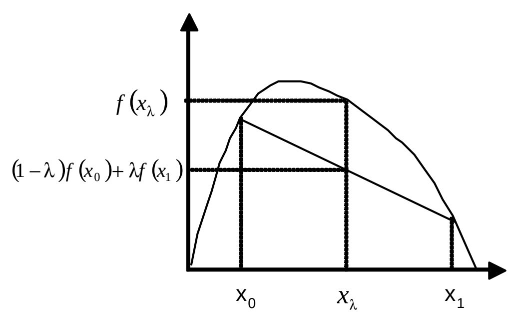

Figure 3: Concave function

Some properties of concave functions are:

1. A concave function lies below any of its tangent lines.
2. The line segment connecting any two points on the curve lies beneath the curve. (As depicted in Figure 3)
3. The slope of the tangent lines are decreasing.

3 Quasi-concave Functions

Closely related to the notion of a concave function is that of a quasi-concave function. It can be shown that the property of concavity of a function is stronger than the property of quasi-concavity in the sense that a function that is concave is automatically quasi-concave. Quasi-concave functions are not always concave, however. (In fact, there are some quasi-concave functions which are convex!)

Formally, the definition of a quasi-concave function is as follows:

Definition 4 *Let x_0 and x_1 be arbitrary points in the domain of a function. Without loss of generality, assume $f(x_0) \leq f(x_1)$. A function f is quasi-concave if for any $\lambda \in [0,1]$, we have that $f(x_\lambda) \geq f(x_0)$ where $x_\lambda = (1-\lambda) x_0 + \lambda x_1$.*

This definition says that if we take the line segment connecting any two points x_0 and x_1 in the domain of the function, the functional value at any point on the line segment is greater than the smaller of the functional values of the two points. That is,

$$f(x_\lambda) \geq \min\{f(x_0), f(x_1)\}$$

An example of a function that is quasi-concave but not concave is the standard bell curve. Due to the tails on the left and the right side, it is not a concave function. (To see this, draw lines which are tangent to the tails and note that they do not lie strictly above the curve) This function is quasi-concave, however. This fact can be demonstrated by taking any two points on the curve and noting that in between those two points, the curve never drops below the lower of those two points.

In one dimension, quasi-concavity of a differentiable function is equivalent to saying that its derivative changes sign no more than once on its domain. It should be noted that a straight line is quasi-concave (as well as concave!) and that this property only holds in one dimension. The multi-dimensional criteria for a function to be quasi-concave are much more complicated.

The notion of quasi-concavity is useful in economics in that one of its properties is the convexity of its upper contour sets. If the function in question is a utility function, then having convex upper contour sets says that the area that is preferred to a given indifference curve is convex. This gives us the nice shaped indifference curves with which we are familiar.