Derived Distributions

$Y = g(X)$
$Z = g(X, Y)$
$(U, V) = g(X, Y)$

$\Big\}$ g fcns are "nice"

$Z = X + Y$

If discrete:

$$P_{UV}(u,v) = \sum_{(x,y):\, g(x,y) = (u,v)} P_{XY}(x,y)$$

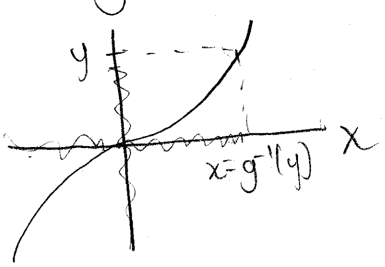
◦ conceptually, nothing deep here.

Find cdf F_Y and differentiate.

$$Y = X^2 \Rightarrow F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$\Rightarrow f_Y(y) = \frac{dF_Y}{dy} = f_X(\sqrt{y}) \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{y}} + f_X(-\sqrt{y}) \cdot \frac{1}{2} \frac{1}{\sqrt{y}}.$$

If g is monotonic: $g \uparrow$



e.g. $g(x) = x^3$

$g(x) \le y \Leftrightarrow x \le g^{-1}(y) = y^{1/3}$

$$F_Y(y) = P(Y \le y) = P(X \le g^{-1}(y)) = F_X(g^{-1}(y))$$

$$f_Y(y) = f_X(g^{-1}(y)) \frac{dg^{-1}}{dy}(y)$$

e.g. $f_Y(y) = f_X(y^{1/3}) \cdot \dfrac{1}{3y^{2/3}}$

$$\frac{dg^{-1}}{dy}\bigg|_y = \frac{1}{\frac{dg}{dx}\Big|_{x=g^{-1}(y)}}$$

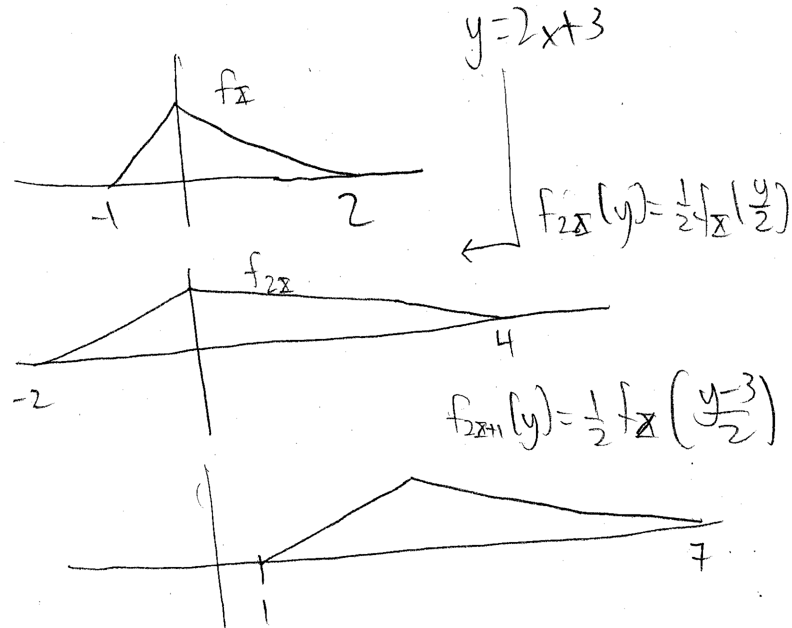
Intuition: $f_X(x)\delta \approx f_Y(y)\cdot \delta \frac{dg}{dx}\Big|_x$

$$\Rightarrow f_Y(y) \approx f_X(g^{-1}(y)) \frac{1}{\frac{dg}{dx}\Big|_{x=g^{-1}(y)}} = f_X(g^{-1}(y)) \frac{dg^{-1}}{dy}\bigg|_y$$

Special Case:

$Y = aX + b = g(X) \Rightarrow \frac{dg}{dX} = a \Rightarrow \frac{dg^{-1}}{dY} = \frac{1}{a}$

$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$



Let $X \sim N(0,1) \Rightarrow f_X(x) = ce^{-x^2/2}$

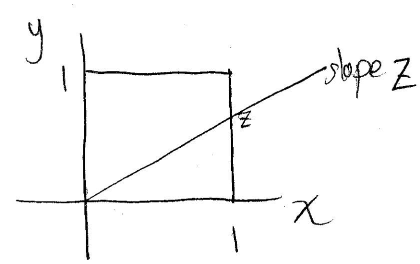
$aX + b = Y \sim \frac{c}{|a|} e^{-\frac{(y-b)^2}{2a^2}} \sim N(b, a^2)$

$Z = g(X, Y)$

$F_Z(z) = P(g(X,Y) \le z)$

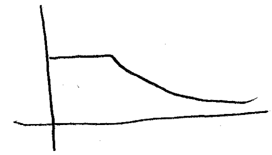
Spse $X, Y \sim$ indep $U[0,1]$ and $Z = Y/X$.

$\Rightarrow f_{X,Y}(x,y) = f_X(x) f_Y(y) = \begin{cases} 1 & 0 \le x, y \le 1 \\ 0 & \text{else} \end{cases}$



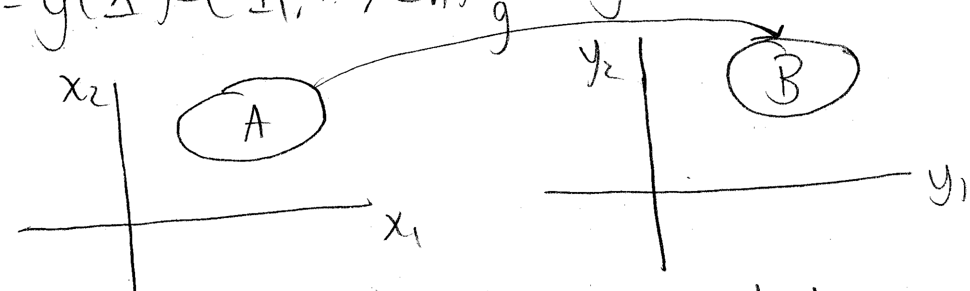
$F_Z(z) = P(Y/X \le z)$

$= \frac{z}{2} \Rightarrow f_Z(z) = \begin{cases} \frac{1}{2} & \text{if } 0 \le z \le 1 \\ \frac{1}{2z^2} & \text{if } z > 1 \end{cases}$



$X = (X_1, \dots, X_n)$

$Y = g(X) = (Y_1, \dots, Y_n)$ $\quad g: \mathbb{R}^n \to \mathbb{R}^n$



assume $g \in C^1$ and g is 1-1

$$\iint_A f_X(x)\, dx = \iint_B f_Y(y)\, dy$$

$$\| $$

$$\iint_B f_X(g^{-1}(y)) \, |J(y)| \, dy, \text{ where } J(y) = \left[\frac{\partial x_i}{\partial y_j}\right]$$

and $|J(y)| = |\det J(y)|$

This must hold $\forall B$, and thus

$$f_Y(y) = f_X(g^{-1}(y)) \underbrace{|J(y)|}_{\text{ratio of volumes}}$$

$$f_X(x)\cdot \mathrm{vol}(A) = f_Y(y)\,\mathrm{vol}(B)$$

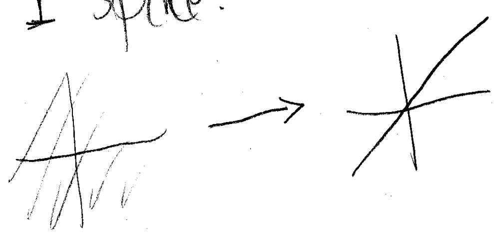
$$\Rightarrow f_Y(y) = f_X(x)\underbrace{\frac{\mathrm{vol}(A)}{\mathrm{vol}(B)}}_{|\det J(y)|} = f_X(g^{-1}(y))\,|\det J(y)|$$

Suppose $\underset{}{Y} = \underset{n\times n}{M}\,X$ $\quad \left[\frac{\partial y}{\partial x}\right]_{ij} = [M]_{ij} \Rightarrow \left[\frac{\partial y}{\partial x}\right] = M$

$\Rightarrow X = M^{-1}Y \quad \left[\frac{\partial x}{\partial y}\right]_{ij} = [M^{-1}]_{ij} \Rightarrow |J(y)| = \det(M^{-1})$

$$f_Y(y) = f_X(M^{-1}y)\,|\det M^{-1}| = \frac{1}{|\det M|}\,f_X(M^{-1}y)$$

What if the mapping is not 1-1? (ie M is singular)
ie. X space gets mapped into a lower dimensional Y space.



Y does not have a density, since it is concentrated on a lower dimensional space

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Suppose
$Y_1 = g(X_1, X_2)$. Cannot use the trick from above
Define $Y_2 = X_2$. Then $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = h(X_1, X_2) = \begin{bmatrix} g(X_1, X_2) \\ X_2 \end{bmatrix}$
and $f_{Y_1}(y_1) = \int f_X(h^{-1}(y_1, y_2))\,|J(y)|\,dy_2$

Suppose $(X, Y) \sim N(0,1)$ independent.

$$f_{XY}(x,y) = \frac{1}{2\pi} \exp\left\{-\frac{(x^2+y^2)}{2}\right\}$$

$$R = \sqrt{X^2+Y^2}$$
$$\theta = \tan^{-1}(Y/X)$$

Let $\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} R\cos\theta \\ R\sin\theta \end{bmatrix} = g^{-1}(R,\theta)$

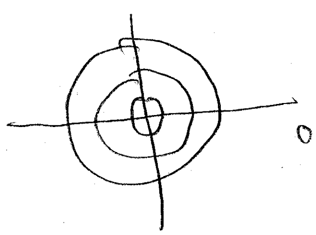
$$f_{R\theta}(r,\theta) = f_{XY}(r\cos\theta, r\sin\theta)\,|J(r,\theta)|$$

$$= \frac{1}{2\pi}\exp\left\{-\frac{r^2}{2}\right\} \cdot \left| \underbrace{\det\begin{bmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{bmatrix}}_{R\cos^2\theta + R\sin^2\theta} \right|$$

$$R(\cos^2\theta + \sin^2\theta) = R$$

$$= \underbrace{\frac{1}{2\pi}} \overbrace{\exp\left\{-\frac{r^2}{2}\right\} \cdot r}^{\text{radii distribution}}, \quad r \geq 0, \ \theta \in [0, 2\pi]$$

$= f_\theta(\theta)\, f_R(r) \Rightarrow R, \theta$ are independent



contour sets

- any angle should be just as likely as any other angle.