Hyperbolic Discounting

$\boxed{b}$ $W_n(k_{T-n}) = \sum_{j=1}^{n} \delta^{n-j} u(c_j(k_{T-n+j}))$

when $u(\cdot)$ is CRRA, we get that

$c_n(k_{T-n}) = (1-s_n)Rk_{T-n}$, $k_{T-j+1} = s_j R k_{T-j}$

$$\Rightarrow W_n(k_{T-n}) = \sum_{j=1}^{n} \delta^{n-j} \frac{((1-s_j)Rk_{T-j})^{1-\sigma}}{1-\sigma}$$

$$= A_n \frac{k_{T-n}^{1-\sigma}}{1-\sigma}$$, since can write $k_{T-j} = f(k_{T-n})$ $\forall j$.

Self $T-n$ solves:

$$\max_{c_{T-n}} \frac{c_{T-n}^{1-\sigma}}{1-\sigma} + \beta\delta \underbrace{W_{n-1}(Rk_{T-n} - c_{T-n})}_{= A_{n-1} \frac{(Rk_{T-n} - c_{T-n})^{1-\sigma}}{1-\sigma}}$$

FOC:

$$(c_{T-n}) = \frac{(\beta\delta A_{n-1})^{-\frac{1}{\sigma}}}{1+(\beta\delta A_{n-1})^{-1/\sigma}} Rk_{T-n}$$

$$\Rightarrow s_n = \frac{1}{1+(\beta\delta A_{n-1})^{-\frac{1}{\sigma}}}$$

Plugging optimal consumption into the objective fcn,

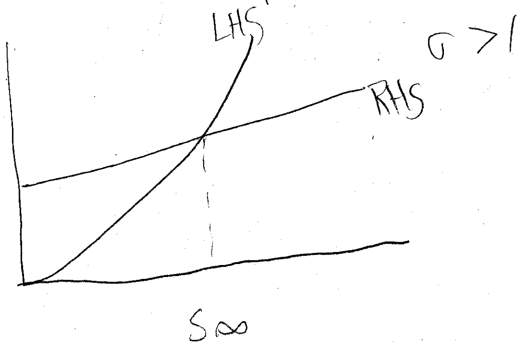
$$W_n(k_{T-n}) = \frac{1}{1-\sigma}\left(\frac{(\beta\delta A_{n-1})^{-\frac{1}{\sigma}}}{1+(\beta\delta A_{n-1})^{-1/\sigma}} Rk_{T-n}\right)^{1-\sigma} + \delta A_{n-1} \cdot \frac{1}{1-\sigma}\left[\frac{1}{1+(\beta\delta A_{n-1})^{-1/\sigma}} Rk_{T-n}\right.$$

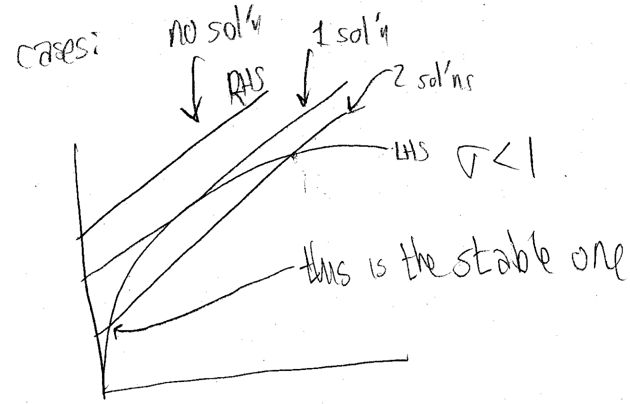
$$= \frac{k_{T-n}^{1-\sigma}}{1-\sigma} \underbrace{\frac{R^{1-\sigma}}{[1+(\beta\delta A_{n-1})^{-1/\sigma}]^{1-\sigma}} \left[(\beta\delta A_{n-1})^{-\frac{1-\sigma}{\sigma}} + \delta A_{n-1} \right]}_{= A_n}$$

$$\Rightarrow A_n = \left(\frac{1-s_{n+1}}{s_{n+1}}\right)^{-\sigma} \frac{1}{\beta\delta}$$

$$\Rightarrow s_{n+1} = \frac{1}{1+[1-s_n][\delta R^{1-\sigma}[\beta(1-s_n)+s_n]]^{-1/\sigma}}$$

c) $s_\infty^\sigma = \delta R^{1-\sigma}(\beta + (1-\beta)s_\infty)$





d) $k_{t+1} = s_\infty R \cdot k_t$ In ss, $k_{t+1} = k_t \Rightarrow s_\infty R^* = 1$
where w is normalized to 0.

$$s_\infty^\sigma = \delta R^{1-\sigma}(\beta + (1-\beta)s_\infty)$$

$$(\underbrace{s_\infty R^*}_{=1})^\sigma = \delta R^* \beta + \underbrace{\delta R^*}_{=1} s_\infty (1-\beta)$$

$$\Rightarrow R^* = \frac{1-\delta(1-\beta)}{\delta\beta}$$

$R^* = f'(k^*) + (1-\delta) \rightarrow$ pins down k^*.

In GE to find steady state, we often do partial equilibrium and look for the right price.

Planning problem

Given $\underbrace{F_0(v_0)}_{\text{distribution of promised utilities}}$, $\overbrace{\tilde{a}_0}^{\text{aggregate capital}}$. Let $y(s_t)$ be efficiency units of labor

$\bar{Y} = \sum y(s)\underbrace{\pi^*(s)}_{\text{invariant distribution}}$

an allocation: $\{c_t(s^t; v_0)\}, \{a_t\}_{t=0}^{\infty}$ a_0 given

Feasibility:

(T) $\int \sum_{s^t} c_t(s^t; v_0) Pr[s^t] dF_0(v_0) + a_{t+1} \leq F(a_t, \bar{Y})$, $t = 0, 1, \ldots$

$\underbrace{(PK)}_{\text{promise keeping}}$ $v_0 = \sum_{t, s^t} \beta^t u(c_t(s^t; v_0)) Pr[s^t]$

(IC) $\sum_{\tau, s^t} \beta^\tau u(c_{t+\tau}(s^{t+\tau}, v_0)) Pr[s^{t+\tau} | s^t] \geq \bar{U}_{AUT}(s_t)$

Efficiency

Given F_0, an allocation $(\{c_t\}, \{a_t\})$ is efficient if it is feasible: (T) + (PK) + (IC) and there does not exist another $(\tilde{F}_0, \{\tilde{c}_t\}, \{\tilde{a}_t\})$ with $\tilde{F}_0 \overset{FOSD}{\succ} F_0$

2 approaches:

1) Find optimization problem
- Lagrangian multipliers $\{Q_t\}$ on (T)

2] Think of a relaxed efficiency criterion (replace (T) with (RT))

- $\{Q_t\}$ "prices"
- Find optimization problem. See if solution is feasible under original problem.

Introduce "relaxed technology":

$$(RT): \quad \sum_{t=0}^{\infty} Q_t \left[\int \sum_{s^t} c_t(s^t, v_0) \Pr[s^t] \, dF_0(v_0) + a_{t+1} - F(a_t) \right] \leq 0$$

for $\{Q_t\}$.

Impose $\sum_{t=0}^{\infty} Q_t < +\infty$, $Q_t > 0$

It can never be true that (RT) holds with strict inequality at optimum

Lemma: Relaxed efficient iff

min (LHS of (RT)) s.t. (PK) + (IC)

$$(a_{t+1}): \quad -Q_{t+1} F'(a_t) + Q_t = 0 \Rightarrow \frac{Q_{t+1}}{Q_t} F'(a_{t+1}) = 1$$

With this approach, can use: $\min \int = \int \min$

$\Rightarrow$ study component planning problems

- simpler way of choosing $\{c_t(s^t, v_0)\}$

$$\Rightarrow \min \sum_{t=0}^{\infty} Q_t \underbrace{c_t(s^t, v_0)}_{\text{interpret this as the firm's problem}} \Pr[s^t] \quad \text{s.t. (PK), (IC)}$$

Claim: Suppose given $\{Q_t\}$ and F_0 optimum for relaxed planning problem: $\{c_t(s^t, v_0), a_t\}$ and (T) holds for all t.

$\Rightarrow (F_0, c_t, a_t)$ is efficient. (cf 1^{st} Welfare Thm)

Reverse is also true. (cf 2^{nd} Welfare Thm)

SS: $F_0, \{c_t\}, \{a_t\}$ with

$\boxed{1}$ a_t constant

$\boxed{2}$ $F_t = F_0$

Note: if a_t constant, $\frac{Q_{t+1}}{Q_t} F'(a^{ss}) = 1$

$$\Rightarrow \frac{Q_{t+1}}{Q_t} = \frac{1}{F'(a_{ss})} \quad \text{(constant ratio)}$$