One more problem set: short.

Hyperbolic Discounting

Phelps/Pollak (68)
Laibson (97)

- infinite horizon: $t=0,1,\ldots$
- one individual who lives forever
  - sequence of different "selfs"
  - self t has preferences: $u(c_t)+\beta\sum_{j=1}^{\infty}\delta^j u(c_{t+j})$, $\beta,\delta\in(0,1]$
    - $\beta=1 \Rightarrow$ exponential (standard) discounting
    - $\beta<1 \Rightarrow$ quasi-hyperbolic discounter
- savings technology $f(k_t)$ (e.g. $f(k_t)=Rk_t$)
  - $c_t+k_{t+1}\le f(k_t)$
  - k_0 given
- "First best" - self 0 decides on all c_t.
  - From period 1 on, solves: $\max \sum_{t=1}^{\infty}\delta^{t-1}u(c_t)$

$$\text{s.t. } c_t+k_{t+1}\le f(k_t) \quad \forall t$$
$$k_1$$

  - Value function: $W(k)=\max_c\{u(c)+\delta W(f(k)-c)\}$

$$\Rightarrow W(k),\ \underbrace{c(k)}_{\text{policy fcn}}$$

  - Period 0: $V(k_0)=\max_{c_0}\{u(c_0)+\beta\delta W(f(k_0)-c_0)\}$

$$\Rightarrow V(k_0),\ \tilde{c}(k_0)$$

- "No commitment" - self t decides on c_t.
  - noncooperative game
  - Focus on Markov equilibrium, symmetric
    - $c_t(k_t)=c(k_t)$ (action depends only on state)

- compute Markov equilibrium
  - For any $c(k)$, define: $W(k_t) = \sum_{j=0}^{\infty} \delta^j u(c(k_{t+j}))$
    - $k_{t+j+1} = f(k_{t+j}) - c(k_{t+j})$
    - ie taking as given that all future selves play the Markov strategy
  - Taking $c^*(k)$ as given, self 0 solves

$$V(k_0) = \max_c \{u(c) + \beta\delta W(k_1)\}$$

$$\text{s.t. } k_1 = f(k_0) - c$$

$$W(k_1) = \sum_{j=0}^{\infty} \delta^t u(c^*(k_{1+j}))$$

$\Rightarrow c^*(k_0)$ (symmetric equilibrium strategy)

$$\Rightarrow V(k_0) = u(c^*(k_0)) + \beta\delta \sum_{j=0}^{\infty} \delta^j u(c^*(k_{1+j}))$$

$$\Rightarrow V(k_0) - (1-\beta)u(c^*(k_0)) = \beta u(c^*(k_0)) + \beta\delta \sum_{j=0}^{\infty} \delta^j u(c^*(k_{1+j}))$$

$$= \beta W(k_0)$$

$$\Rightarrow \beta W(k) = \max_c \{u(c) + \beta\delta W(f(k) - c)\} - (1-\beta)u(c^*(k))$$

Steps:

- start with $c^*(k)$ that solves (for a guessed $W(k)$)
- $TW(k) = \frac{1}{\beta} \max_c \{u(c) + \beta\delta W(f(k) - c)\} - \left(\frac{1-\beta}{\beta}\right) u(c^*(k))$

$\Rightarrow$ Fixed point: $W = TW$

$\Rightarrow$ The associated $c^*(k)$ is a symmetric Markov equilibrium of this game.

Example

assume
- $u(c) = \log(c)$
- $f(k) = Ak^\alpha$ (ie if $\alpha = 1$, $R = A$)

Guess: $W(k) = a \log k + b$

$$\beta T W(k) = \max_c \{ \log c + a\beta\delta \log(Ak^\alpha - c) + \beta\delta b \} - (1-\beta)\log(c^*(k))$$

FOC: $\frac{1}{c} = \frac{a\beta\delta}{Ak^\alpha - c} \Rightarrow c = \frac{1}{1+a\beta\delta} Ak^\alpha \overset{\text{symmetry}}{\Rightarrow} c^* = \frac{1}{1+a\beta\delta} Ak^\alpha$

$$\Rightarrow \beta(a\log(k) + b) = \log\left(\frac{Ak^\alpha}{1+a\beta\delta}\right) + a\beta\delta \log\left(\frac{a\beta\delta}{1+a\beta\delta} Ak^\alpha\right) + \beta b\delta$$
$$- (1-\beta)\log\left(\frac{1}{1+a\beta\delta} Ak^\alpha\right)$$

Solving for a, b, we get: (we don't really care about b)

$$\beta a = \alpha(1 + a\beta\delta - (1-\beta)) \Rightarrow \beta a - \alpha\beta\delta a = \alpha\beta \Rightarrow a = \frac{\alpha}{1-\alpha\delta}$$

$$\Rightarrow c^*(k) = \frac{1-\alpha\delta}{1-\alpha\delta(1-\beta)} Ak^\alpha$$ save less than when $\beta = 1$ (exponential discounters)

Dynamic adverse Selection (Recursive Contracts) Atkeson-Lucas

- micro-founding the lack of complete insurance
- risk averse agent subject to taste shocks (unobservable)
  $\theta_t u(c_t)$ (θ_t - privately known)
- risk-neutral principal who provides consumption that may vary
- $u(c) = \ln c$, $E[\theta] = 1$

FB: $$\max_{c_t(\theta^t)} \sum_{t,\theta^t} \beta^t \theta_t u(c_t(\overset{\text{history of taste shocks}}{\theta^t})) \Pr[\theta^t]$$

$$\text{s.t.} \sum_{t,\theta^t} q^t c_t(\theta^t) \Pr[\theta^t] \le R \qquad (\lambda)$$

FOC: assuming $(\beta = q)$ $\theta_t u'(c_t(\theta^t)) = \lambda$

∘ c_t is just a fcn of θ_t, not of θ^{t-1}.

Second Best:

$$K(w_0) = \min \sum_{t,\theta^t} q^t c(u_t(\theta^t)) \Pr[\theta^t] \quad , c(u) = \exp\{u\}$$

$$\text{s.t. } W_0 = \sum_{t,\theta^t} \beta^t \theta_t u_t(\theta^t) \Pr[\theta^t] \quad \text{(Promise-Keeping)}$$

$$\sum_{t,\theta^t} \beta^t \theta_t u_t(\theta^t) \Pr[\theta^t] \geq \sum_{t,\theta^t} \beta^t \theta_t u_t(\overbrace{\Delta^t(\theta^t)}^{\text{reporting fcn}}) \Pr[\theta^t] \quad \text{(IC)} \quad \forall \Delta^t$$

where $\Delta^t : \underbrace{\Theta \times \ldots \times \Theta}_{t+1 \text{ time}} \rightarrow \underbrace{\Theta \times \ldots \times \Theta}_{t+1 \text{ times}}$

induces: $\Delta_t : \Theta^{t+1} \rightarrow \Theta$

∘ truth-telling: $\Delta^{t*}(\theta^t) = \theta^t$, $\Delta^*_t(\theta^t) = \theta_t$

assume iid shocks: $\Pr[\theta^t] = \pi(\theta_t)\pi(\theta_{t-1}) \cdots \pi(\theta_0)$

∘ continuation utility

∘ $W_t(\theta^{t-1}) = \sum_{s,\theta^{t+s}} \beta^s \theta_s u_{t+s}(\underbrace{\theta^{t+s}}) \Pr[\theta^s]$

must coincide with θ^{t-1} for first $t-1$ periods.

$$\text{(PK)} \Rightarrow W_t(\theta^{t-1}) = \sum_{\theta^t} [\theta_t u_t(\theta^{t-1}, \theta_t) + \beta w_{t+1}(\theta^{t-1}, \theta_t)] \pi(\theta_t)$$

$\forall t, \theta^{t-1}$

using one-shot deviation principle

$$\text{(IC)} \Rightarrow \theta_t u_t(\theta^{t-1}, \theta_t) + \beta w_{t+1}(\theta^{t-1}, \theta_t)$$
$$\geq \theta_t u_t(\theta^{t-1}, \hat{\theta}_t) + \beta w_{t+1}(\theta^{t-1}, \hat{\theta}_t) \quad \forall \hat{\theta}_t \in \Theta, \forall t, \forall \theta^t$$

Recursive problem: $(q=\beta)$

$$K(w) = \min_{u(\theta),\, w'(\theta)} \sum_\theta \left[c(u(\theta)) + \beta K(w'(\theta)) \right] \pi(\theta)$$

$$\text{s.t. } w = \sum_\theta \left[\theta u(\theta) + \beta w'(\theta) \right] \pi(\theta) \qquad (PK)$$

$$\theta u(\theta) + \beta w'(\theta) \geq \theta u(\hat\theta) + \beta w'(\hat\theta) \quad \forall \theta, \hat\theta \qquad (IC)$$

will get $\left.\begin{matrix} u(w,\theta) \\ w'(w,\theta) \end{matrix}\right\}$ policy functions

Recall that $u(c) = \ln c$

Suppose $w_0 = 0$. Get $u_t^*(\theta^t)$

Claim: $w_0 \neq 0 \Rightarrow$ get $\hat u_t(\theta^t) = u_t^*(\theta^t) + (1-\beta) w_0$ optimal $\forall t, \theta^t$

Then $K(w_0) = \sum_{t,\theta^t} \beta^t \exp\{u_t^*(\theta^t) + (1-\beta)w_0\} \Pr[\theta^t]$

$$= \exp\{(1-\beta)w_0\} \underbrace{\sum_{t,\theta^t} \beta^t \exp\{u_t^*(\theta^t)\} \Pr[\theta^t]}_{=A}$$

$\Rightarrow K(w_0) = A \exp\{(1-\beta)w_0\}$

also, know:

$g^u(\theta, w) = (1-\beta)w + h^u(\theta) \rightarrow$ random walk

$g^{w'}(\theta, w) = w + h^{w'}(\theta) \rightarrow$ random walk

iid shocks

$\Rightarrow w_t(w_0, \theta^t) = w_0 + \sum_{s=0}^{t-1} h^{w'}(\theta_s) + h^{w'}(\theta_t)$

$c = \exp\{u\} \rightarrow$ geometric random walk

$\Rightarrow u_t(w_0, \theta^t) = (1-\beta)w_0 + (1-\beta)\sum_s h^{w'}(\theta_s) + (1-\beta)h^{w'}(\theta_t) + h^u(\theta_t)$

$$\mathrm{Var}(u_t(w_0, \theta^t)) = \mathrm{Var}((1-\beta)w_0 + (1-\beta)^2 \cdot t\, \mathrm{Var}(h^{w'}(\theta)) + \mathrm{var}(h^u(\theta)) \rightarrow +\infty \text{ as } t \to \infty$$

$E_t[c_{t+1}] = c_t \Rightarrow c_t \to 0$ a.s. "Immiseration result"

Martingale convergence thm