CARA: had a nice closed form.

Consumption function: $g^c(a,s) = ra + \frac{r}{R} y(s) + \tilde{\alpha}(s)$

◦ if future is risky, more precautionary savings.

also: $g^{a'}(a,s) = (1+r)a + y(s) - g^c(a,s)$

$$= a + \underbrace{\left(1 - \frac{r}{R}\right)}_{= \frac{1}{1+r}} y(s) - \hat{\alpha}(s)$$

$$= a + \frac{1}{R} y(s) - \hat{\alpha}(s)$$

◦ "if I ever own more assets, I forever own more assets"

In iid case: $\hat{\alpha}(s) = \hat{\alpha}$ and $y(s)$ is iid.

$\Rightarrow a' - a = \frac{1}{R} y(s) - \hat{\alpha}$ $\Rightarrow a_t$ is a random walk

$$E_t[a_{t+T}] = \frac{1}{R} E\left[\sum_{t=t+1}^{T} y_t(s)\right]$$

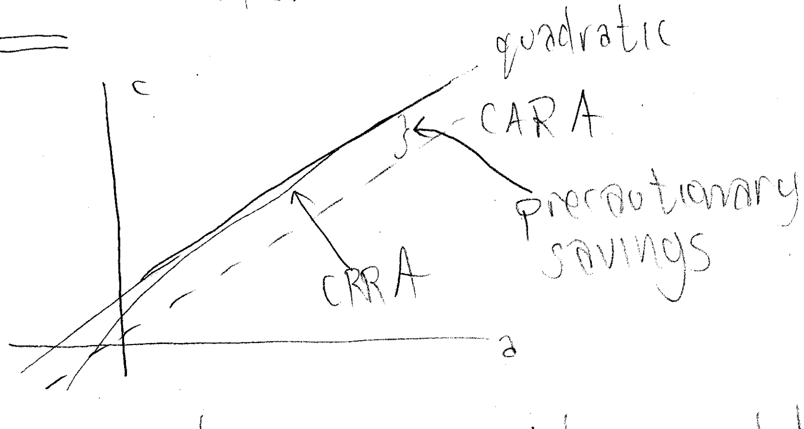
$$Var_t(a_{t+T}) = \frac{1}{R^2} \sum_{t=t+1}^{T} Var_t(y_t(s))$$

If everyone starts off equally, then cross-sectional variance grows linearly. (Inequality explodes.)

Imagine everyone has $a_{i0} = a_0$

$$\Rightarrow F_0(a) = \begin{cases} 1 & a \geq a_0 \\ 0 & a < a_0 \end{cases}$$

- This implies an equation for $\{F_t(a)\}_{t=0}^{\infty}$, the stochastic process defining the distribution of assets over time.
- Does $F_t \xrightarrow{d} F$?
  - In the previous model, the variance grows linearly, so the answer is no.
- Is there some F_0 s.t. $F_t = F_0 \; \forall t$?
  - also no
- This gives us motivation for studying a different model.



- By CARA assumption, wealth does not affect the way lotteries are viewed

Two changes to the model:
- Borrowing constraints (non-negative consumption)
- CRRA - smaller precautionary effect when you are rich than when you are poor

CRRA:

- iid income
- $c_t \geq 0$
- borrowing constraints

Borrowing constraints:

$$c_t + a_{t+1} \leq R a_t + y_t \quad , \quad a_t \text{ risk free}$$

No Ponzi: $R^{-t} a_t \to 0$

$$c_t \geq 0$$

$$\Rightarrow \sum_{\tau=0}^{\infty} R^{-\tau} y_{t+\tau} + R a_t \geq \sum_{\tau=0}^{\infty} c_{t+\tau} R^{-\tau} \quad \text{for each } t, s^t.$$

$$\Rightarrow a_t \geq -\frac{y_{min}}{r}$$

(ie set $c_{t+\tau} = 0 \ \forall \tau$ and $y_{t+\tau} = y_{min} \ \forall \tau$)

- can't borrow more than you are sure to be able to pay off.

Consumption will depend on cash-in-hand:

- $x = Ra + y$
- $a_t \geq -\varphi$ $\qquad \varphi = \min\{\frac{y_{min}}{r}, b\}$

Bellman equation

$$v(x) = \max_{a' \geq \varphi} \{u(x-a') + \beta E[v(Ra' + y(s'))]\}$$

Let $\hat{a} = a + \varphi$, $z = R\hat{a} + y - R\varphi$

$\Rightarrow \quad z_t = \hat{a}_t + c_t$

and $\hat{a} \geq 0$

Thus, $$v(z) = \max_{\hat{a}' \geq 0} \{u(z - \hat{a}') + \beta E[v(\overbrace{R\hat{a}' + \underbrace{y(s') - r\varphi}_{\text{can think of this as income process: } \hat{y}(s')}}^{z'})]\}$$

Now, let us change notation so that we have no hats. (ie it is wlog that $\varphi = 0$)

$$v(z) = \max_{a' \geq 0} \{u(z-a') + \beta E[v(Ra' + y(s'))]\}$$

Suppose $\beta R > 1$

Euler equation: $u'(c_t) \geq \beta R \, E_t[u'(c_{t+1})]$

$$u'(c_t) \geq \frac{1}{\beta R} u'(c_t) \geq E_t[u'(c_{t+1})]$$

$E_t[u'(c_{t+\tau})] \to 0$ as $\tau \to \infty$

By the martingale convergence theorem (MCT)

$$u'(c_t) \to 0 \quad a.s.$$

Suppose $\beta R = 1 \Rightarrow u'(c_t) = E_t[u'(c_{t+1})]$

MCT: $u'(c_t) \to \underbrace{\tilde{m}}_{\text{random variable}}$ a.s. with $E[|\tilde{m}|] < +\infty$

(*) a martingale is a generalized random walk.

Want to show $\tilde{m} = 0$ a.s. (ie $\tilde{m}(s^\infty) = 0$ a.a. s^∞)

Fix s^∞ and spse $m > 0 \Rightarrow u'(c_t) \to m > 0$

s.t. y_t does not converge $\Rightarrow c_t \to \bar{c} = (u')^{-1}(m)$

By Envelope condition, $V'(z_t) = u'(c_t)$

$$\Rightarrow z_t \to \bar{z}$$

$$\Rightarrow \bar{c} + \bar{a} = R\bar{a} + y_t \quad \rightarrow\leftarrow$$

Thus, $c_t \to \infty$ a.s. and hence $a_t \to \infty$ a.s.
(since $u'(c_t) \to 0$)

In the precautionary savings world, $\beta R = 1$ is no longer the knife edge case it used to be.

Suppose $\beta R < 1$. Three properties

1] $g^{c}(z)$ increasing } both are continuous.

2] $g^{a'}(z)$ increasing

3] a_t bounded above.

assume $-\frac{u''(c)}{u'(c)} \to 0$ as $c \to \infty$ (rules out behavior in CARA case)

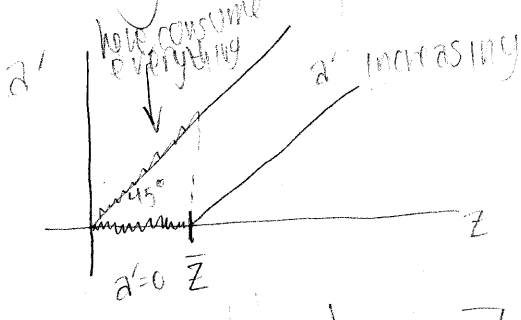
Recall Bellman equation:

$$v(z) = \max_{a' \geq 0} \{ u(z - \hat{a}') + \beta \underbrace{E[v(Ra' + y(s) - r\varphi)]}_{= \Phi(a')}$$

$$v(z) = \max_{a' \geq 0} \{ u(z - a') + \Phi(a') \} \Rightarrow u'(z - a') = \Phi'(a')$$

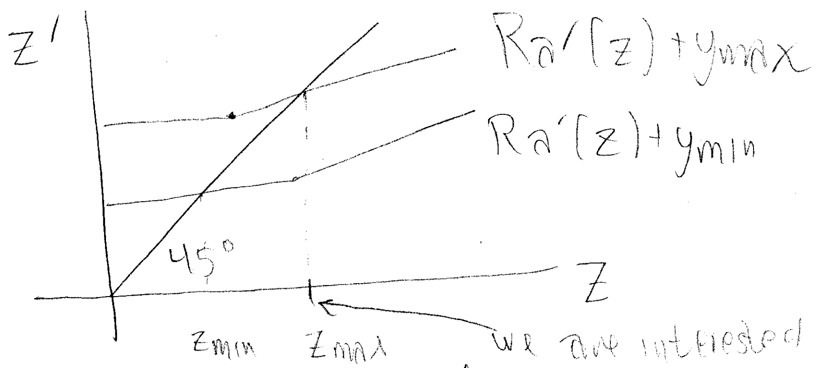
$$z \uparrow \Rightarrow a' \uparrow$$

• very simple problem



• This establishes 1] and 2]

Graphically, this is what is going on.



$z' = Ra' + y + r\varphi$

How do we prove this?

Let $z_{max}'(z) = Ra'(z) + y_{max}$. Want to show for z high enough s.t. $z_{max}'(z) \leq z$.

For high enough z, Euler equation holds w/ equality.

Thus, $u'(c(z)) = \beta R E[u'(c(z'))]$

$\beta R < 1$ makes $u'(c(z')) \uparrow$ in z, and hence c decreases in z.

Idea: Divide through:

$$u'(c(z)) = \beta R E\left[\frac{u'(c(z'))}{u'(\bar{c}(z))}\right] u'(\bar{c}(z))$$

where $\bar{c}(z) = c(z_{max}'(z))$.