## Income fluctuation problem

$$\max_{\{c_t, a_{t+1}\}} E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right] \quad \text{s.t. } c_t + a_{t+1} \le R a_t + w \underbrace{y_t}_{\text{stochastic income}}$$

$$c_t \ge 0, \; a_t \ge 0 \;\; \forall t$$

assume $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$

$y_t$ is a stochastic process

e.g. • $y_t$ iid

• $0 < y^1 < y^2 < \dots$

• Markov: $\Pr[y_t = y^i \mid y_{t-1} = y^j] \equiv P_{ji}$

Given a specific stochastic process for income, how does a consumer make optimal decision?

Bellman equation:

$$V(a, y) = \max_{c, a'} \{ u(c) + \beta E[V(a', y') \mid y] \}$$

$$\text{s.t. } c + a' \le Ra + wy, \; c \ge 0, \; a' \ge 0.$$

solution $\Rightarrow$ $c(a,y)$, $a'(a,y)$ policy functions

How to solve this numerically?

Suppose $y \in \{y_1, y_2\}$

Then $V(a,y)$ is just a matrix (assuming a finite grid for the values of $a$)

Start with $V_0(a,y) \equiv 0$. Given this, solve:

$$V_1(a,y) = \max_{a'} \{ u(Ra + wy - a') + \beta E[V_0(a', y') | y] \}$$

for each

$$= \max_{a'} \left\{ \frac{(Ra - y - a')^{1-\sigma}}{1-\sigma} + \beta [p_\ell(y) V_0(a', y_\ell) + p_h(y) V_0(a', y_h)] \right\}$$

for each $a, y$

where $p_\ell(y) = Pr[y' = \ell | y = y]$

Then recursively solve

$$V_{t+1}(a,y) = \max_{a'} \{ u(Ra + wy - a') + \beta E[V_t(a', y') | y] \}$$

iterate until convergence and save $a'$ matrix:

$\Rightarrow a'(a,y)$, $c(a,y) = Ra + wy - a'(a,y)$

Simulation

1. Draw $(y_1, \dots, y_T)$ vector of income shocks
2. Start with $a_0 = 0$

$$\left. \begin{aligned} a_1 &= a'(0, y_0) \\ a_2 &= a'(a_1, y_1) \\ &\vdots \\ a_T &= a'(a_{T-1}, y_{T-1}) \end{aligned} \right\} (a_1, \dots, a_T)$$

3. average assets $\bar{a} = \frac{1}{T}\sum_{t=0}^{T} a_t$

4. General equilibrium
- $\bar{a}(\beta, \sigma, R, w)$
- $F(K,L) = K^{\alpha}L^{1-\alpha}$ $\quad L = 1$
- $R = 1 + \alpha K^{\alpha-1} - \delta$
- $w = (1-\alpha)K^{\alpha}$
- $K = \left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}}$ $\quad r = \alpha K^{\alpha-1} - \delta$
- $w = (1-\alpha)\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}$

$\Rightarrow \bar{a}(r, w, \sigma, \beta) = K(r, \delta, \alpha)$

$\Rightarrow w(r)$. Find $r$ that equates these two.