Asset Pricing

The simple model has a difficult time explaining the facts: equity premium puzzle.

$$\max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t(s^t)) \Pr[s^t]$$

$$\text{s.t.} \quad c_t(s^t) + \overbrace{q_t(s^t)}^{\text{vector}} \cdot a_{t+1}(s^t) \le \overbrace{W_t(s^t)}^{\text{time } t \text{ wealth}}$$

$$W_{t+1}(s^{t+1}) = \underbrace{y_{t+1}(s^{t+1})}_{\text{income}} + \left(\underbrace{q_{t+1}(s^{t+1})}_{\text{asset price}} + \underbrace{d_{t+1}(s^{t+1})}_{\text{dividend stream}}\right) \cdot a_{t+1}(s^t)$$

Euler equation:

$$u'(c_t) q_t^i = \beta E_t[u'(c_{t+1}) \underbrace{(q_{t+1}^i + d_{t+1}^i)}_{\equiv R_{t+1}^i}]$$

$$= \beta E_t[u'(c_{t+1}) R_{t+1}^i]$$

$$\Leftrightarrow 1 = E_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1}^i\right]$$

transversality condition: $\lim_{j \to \infty} \beta^j E_0[u'(c_{t+j}) q_{t+j} a_{t+j}] = 0$

Pricing formula obtained by repeated substitution:

$$q_t^i = \sum_{j=1}^{\infty} \beta^j E_t\left[\frac{u'(c_{t+j})}{u'(c_t)} d_{t+j}^i\right] \quad \forall i \text{ (assets)}$$

- no bubbles ($\Rightarrow$ by $s_t = 1$ and transversality)

- Hall: Do not need to solve entire model to test the theory. Can test certain implications of intermediate steps. (ie test the Euler equation.)
  - GMM was an immediate response to this.

CCAPM

(1) ∘ $1 = \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} R^i_{t+1}\right]$ This is a special case of:

∘ $1 = E_t[\underbrace{m_{t+1}}_{\text{stochastic discount factor}} R^i_{t+1}]$

$$(1) \iff 1 = \beta E_t\left[\frac{u'(c_{t+1})}{u'(c_t)}\right] E_t[R^i_{t+1}] + \underbrace{\beta \, cov_t\left(\overbrace{\frac{u'(c_{t+1})}{u'(c_t)}}^{\downarrow \text{ in } c_{t+1}}, R^i_{t+1}\right)}_{=0 \text{ if risk-free } i}$$

$R^i_{t+1} = \bar{R}_{t+1} \ \forall t$

- will an asset with a variance command a higher rate of return?
  - depends on the covariance term.
  - if the returns are orthogonal to MRS, then asset will have risk-free rate

- assume log-normal error
- preferences: $u' = c^{-\eta}$

$$\frac{c_{t+1}}{c_t} = \bar{c}_\Delta \exp\left\{\varepsilon_c - \frac{1}{2}\sigma_c^2\right\}$$

- we are parameterizing this

$$\varepsilon_c \sim N(0, \sigma_c^2)$$

$$\Rightarrow E\left[\frac{c_{t+1}}{c_t}\right] = \bar{c}_\Delta$$

(since if $X \sim N(\mu, \sigma^2)$, $E[\exp\{X\}] = \exp\{\mu + \frac{\sigma^2}{2}\}$)

- returns: $R^i = (1+\bar{r}^i)\exp\{\varepsilon_i - \frac{1}{2}\sigma_i^2\}$

$$\varepsilon_i \sim N(0, \sigma_i^2)$$

$$\Rightarrow E[R^i] = \bar{R}^i = 1 + \bar{r}^i$$

- Let $\left(\frac{c_{t+1}}{c_t}, R^i\right)$ be jointly log-normal with covariance σ_{ic}

Euler equation:

$$1 = \beta E_t\left[R_i\left(\frac{c_{t+1}}{c_t}\right)^{-\eta}\right]$$

$$= \beta(1+\bar{r}_i)(\bar{c}_\Delta)^{-\eta} E_t\left[\exp\left\{\varepsilon_i - \frac{1}{2}\sigma_i^2 - \eta\varepsilon_c + \eta\frac{1}{2}\sigma_c^2\right\}\right]$$

$$= \beta(1+\bar{r}^i)(\bar{c}_\Delta)^{-\eta} \underbrace{\exp\left\{(1+\eta)\eta\frac{1}{2}\sigma_c^2 - \eta\sigma_{ic}\right\}}_{\text{by log-normal assumption}}$$

Taking logs:

$$\log(1+\bar{r}^i) = -\log\beta + \gamma\log\bar{c}_\Delta - (1+\gamma)\gamma\tfrac{1}{2}\sigma_c^2 + \gamma\sigma_{ic}$$

$$\Rightarrow \text{risk-free } \bar{r}^f \approx \log(1+\bar{r}^f) = -\log\beta + \gamma\log\bar{c}_\Delta - (1+\gamma)\gamma\tfrac{1}{2}\sigma_c^2$$

$$\text{stock } \bar{r}^s \approx \log(1+\bar{r}^s) = -\log\beta + \gamma\log\bar{c}_\Delta - (1+\gamma)\gamma\tfrac{1}{2}\sigma_c^2 + \gamma\sigma_{sc}$$

$$\underbrace{\bar{r}^s - \bar{r}^f}_{\approx 6\% \text{ empirically}} \approx \log(1+\bar{r}^s) - \log(1+\bar{r}^f) = \gamma\underbrace{\sigma_{sc}}_{\approx 0.219\%}$$

$\Rightarrow \gamma \approx 27$

- either need γ very high or low risk-free rate. This is a puzzle.

Quick primer on GMM:

$$0 = E_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}(R^s_{t+1} - R^f_{t+1})\right]$$

$$0 \approx \frac{1}{T}\sum_{t=1}^{T}\left(\beta\frac{u'(c_{t+1})}{u'(c_t)}(R^s_{t+1} - R^f_{t+1})\right) = \frac{1}{T}\sum_{t=1}^{T}\left(\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}(R^s_{t+1} - R^f_{t+1})\right.$$

also have another moment condition:

$$0 \approx \frac{1}{T}\sum_{t=1}^{T}\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}R^f_{t+1} - 1$$

Choose β, γ to make these close to zero.

Cochrane-Hansen

- Two periods - t, $t+1$ (now and then)
- J "fundamental" assets
  - x^j payoff "then" $x = \begin{bmatrix} x^1 \\ \vdots \\ x^J \end{bmatrix}$
  - q^j price "now" $q = \begin{bmatrix} q^1 \\ \vdots \\ q^J \end{bmatrix}$

- payoff space: $\underbrace{P}$ $= \{p : p = c \cdot x$ for some $c \in \mathbb{R}^n\}$
  space of random variables can get as linear combinations of assets

- pricing function $\pi(p) : P \to \mathbb{R}$
  - $\pi(x^j) = q^j \quad \forall j$.
  - $\pi(c \cdot x) = c \cdot \pi(x) = c \cdot q$ (ie pricing function is linear - Law of one price)
  - if there are two ways of getting same payoff, must cost the same
    $c \cdot x = c \cdot x' \Rightarrow c \cdot q = c \cdot q'$.

Definition: discount factor $y \in P$ (random variable)
$$\pi(p) = E[\underbrace{y}_{\text{stochastic discount factor}} p]$$

By the Riesz representation theorem, $\exists$ such a discount factor.

- Let Y be the set of all such discount factors.

Defn: (No arbitrage) (NA):

$$p \geq 0 \Rightarrow \pi(p) \geq 0$$

$$p > 0 \text{ with positive probability} \Rightarrow \pi(p) > 0$$

NA $\Leftrightarrow$ there is a strictly positive discount factor $y > 0$

- Let Y^{++} be the set of all discount factors.

examples: $m = \beta \frac{u'(c_{t+1})}{u'(c_t)}$

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all theories are saying:

- $q = E[mp]$
- $m = f(\text{data, parameters})$