Question 2:

Define $G(c_t, V_t, V_{t+1}) = V_t - [(1-\beta)c_t^{\rho} + \beta(E_t[V_{t+1}^{\alpha}])^{\rho/\alpha}]^{1/\rho}$

Then $G(\lambda c_t, \lambda V_t, \lambda V_{t+1}) = \lambda G(c_t, V_t, V_{t+1})$

Thus, if (c_t, V_t, V_{t+1}) solves $G(c_t, V_t, V_{t+1}) = 0$, then $(\lambda c_t, \lambda V_t, \lambda V_{t+1})$ solves $G(\lambda c_t, \lambda V_t, \lambda V_{t+1}) = 0$.

Next, $(4, 4, 4)$ solves $G(4,4,4) = 0$ as well.

Welfare Theorems in General Commodity Spaces

14.123: Finite number K of different goods.

- commodity space $X = \mathbb{R}^K$
- $\forall k$, $p_k \in \mathbb{R}$ is the price of good k
- $p \in \mathbb{R}^K$ is a price system
- Value of $x \in X$ is $\langle p, x \rangle = p \cdot x = \sum_{k=1}^{K} p_k x_k$

Want to generalize this:

- e.g. infinite horizon model: $(c_0, c_1, c_2, \dots)$
- e.g. uncertainty $(c_{s_1}, c_{s_2}, \dots)$

Now, assume the commodity space is the vector space S with some norm $\|\cdot\|$

For example,

$S = \{\{c_t\}_{t=0}^{\infty}\}$ is the space of infinite sequences

$\|\cdot\| = \|\cdot\|_{\infty}$ (sup norm): $\|\{c_t\}_{t=0}^{\infty} - \{\bar{c}_t\}_{t=0}^{\infty}\| = \sup_t |c_t - \bar{c}_t|$

I consumers: $i=1,\dots,I<+\infty$

Consumer i chooses among points in $X_i \subseteq S$.

- preferences: $u_i: X_i \to \mathbb{R}$

production: J firms: $j=1,\dots,J$.

Each firm chooses a production vector in $Y_j \subseteq S$.

an <u>allocation</u> is an $(I+J)$-tuple: $[(x_i),(y_j)]$

an allocation $[(x_i),(y_j)]$ is <u>feasible</u> if $\forall i,j$,

$x_i \in X_i$ and $y_j \in Y_j$ and

$\sum_{i=1}^{I} x_i - \sum_{j=1}^{J} y_j = 0$

an allocation $[(x_i),(y_j)]$ is <u>Pareto optimal</u> if $[(x_i),(y_j)]$ is feasible and there does not exist another feasible allocation $[(x_i'),(y_j')]$ such that $u_i(x_i') \geq u_i(x_i)$ $\forall i$ and $u_i(x_i') > u_i(x_i)$ for some i.

a <u>price system</u> is a function $\varphi: S \to \mathbb{R}$ satisfying

1] $\varphi(\alpha x + \beta y) = \alpha \varphi(x) + \beta \varphi(y)$ $\quad \forall x,y \in S$ $\forall \alpha, \beta \in \mathbb{R}$.

2] If $\|x_n - x\| \to 0$, then $|\varphi(x_n) - \varphi(x)| \to 0$

(ie a price system is a continuous linear functional.)

a competitive equilibrium is an allocation $[(x_i^0),(y_j^0)]$ and a price system φ satisfying

1] $[(x_i),(y_j)]$ is feasible

2] For each i, for each $x \in X_i$ with $\varphi(x) \leq \varphi(x_i^0)$,
$u_i(x) \leq u_i(x_i^0)$

3] For each j, for each $y \in Y_j$, $\varphi(y) \leq \varphi(y_j^0)$

Theorem: (First welfare theorem) Suppose that $\forall i$ and $\forall x \in X_i$, there exists a sequence $\{x_n\}$, $x_n \in X_i$ s.t. $\|x_n - x\| \to 0$ and $u_i(x_n) > u_i(x)$ $\forall n$. (local non-satiation).

Then, if $[(x_i^0),(y_j^0),\varphi]$ is a competitive equilibrium, $[(x_i^0),(y_j^0)]$ is Pareto optimal

Pf: Let $[(x_i^0),(y_j^0),\varphi]$ be a CE. Then

1] for all i, $u_i(x) = u_i(x_i^0) \Rightarrow \varphi(x) \geq \varphi(x_i^0)$

Suppose 1] does not hold. Then $\exists\, i$ s.t. $u_i(x) = u_i(x_i^0)$ but $\varphi(x) < \varphi(x_i^0)$. Let $\{x_n\}$ be a sequence with $x_n \in X_i$ s.t. $\|x_n - x\| \to 0$ and $u_i(x_n) > u_i(x) = u_i(x_i^0)$ (by non-satiation)

By the continuity of the price system,

if $\|x_n - x\| \to 0$, $|\varphi(x_n) - \varphi(x)| \to 0$. Thus, $\forall n$ sufficiently large, $\varphi(x_n) < \varphi(x_i^0)$

Thus, $\exists x_N$ s.t. $u_i(x_N) > u_i(x_i^0)$ and $\varphi(x_n) < \varphi(x_i^0)$ for all i, but this contradicts point $\boxed{2}$ of the defn of CE.

$\boxed{2}$ Suppose there exists a feasible allocation $[(x_i'),(y_j')]$ s.t. $u_i(x_i') \geq u_i(x_i^0)$ $\forall i$ and $u_i(x_i') > u_i(x_i^0)$ for some i.

Since $[(x_i'),(y_j')]$ is feasible, $\forall i$, $u_i(x_i') > u_i(x_i^0)$ by point $\boxed{2}$ of the defn of CE.

Summing up, $\sum_{i=1}^{I} \varphi(x_i') > \sum_{i=1}^{I} \varphi(x_i^0) \Rightarrow \varphi(\sum_{i=1} x_i') > \varphi(\sum_{i=1} x_i^0)$

linearity of price system

Finally, $\sum_{j=1}^{J} \varphi(y_j') = \varphi(\sum_{j=1}^{J} y_j') = \varphi(\sum_{i=1}^{I} x_i')$

$> \varphi(\sum_{i=1}^{I} x_i^0) = \varphi(\sum_{j=1}^{J} y_j^0) = \sum_{j=1}^{J} \varphi(y_j^0)$,

but this contradicts point $\boxed{3}$ of the defn of CE, since there must be some j^* such that $\varphi(y_{j^*}') > \varphi(y_j^0)$.

Theorem (second-welfare): assume:

1] $\forall i$, X_i is convex

2] $\forall i$, if $x, x' \in X_i$ and $u_i(x) > u_i(x')$, $\theta \in (0,1)$, then $u_i(\theta x + (1-\theta)x') > u_i(x')$ (quasiconcavity)

3] The set $Y = \sum_{j=1}^{J} Y_j = \{\bar{y} \in S : \bar{y} = \sum y_j, \ y_j \in Y_j\}$ is convex

"Hahn-Banach" Theorem generalizes the separating hyperplane theorem for a general vector space.