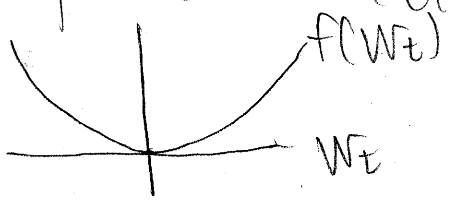
Stochastic calculus, cont.

1. Random process, $X \Leftrightarrow \mathcal{F}_t^X$ - filtration generated by X
2. Brownian motion Z_t
   - $Z_{t+s} - Z_t | \mathcal{F}_t \sim N(0, s)$
3. Stochastic integral
4. Ito's Lemma
   - Diffusion process: $dW_t = \underbrace{\overset{\text{expected rate of change}}{\mu_t}}_{\text{drift}} dt + \underbrace{\beta_t}_{\text{volatility}} dZ_t$
   - Ito's lemma: If $f \in C^2$, and W_t is a diffusion process, then $f(W_t)$ is a diffusion process and

$$df(W_t) = \left(f'(W_t)\mu_t + \frac{1}{2}\beta_t^2 f''(W_t)\right)dt + \beta_t f'(W_t)\,dZ_t$$

   - If there is curvature, then in expectation the process will drift in that function:



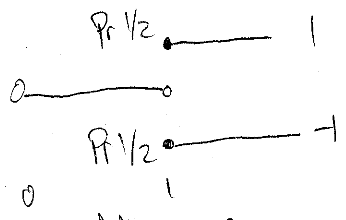
- $f(W_t, t) \Rightarrow df(W_t, t) = \left(f_w(W_t, t)\mu_t + \frac{1}{2}\beta_t^2 f_{ww}(W_t, t) + f_t(W_t, t)\right) dt + \beta_t f_w(W_t, t)\,dZ_t$

5] Martingale representation theorem

- Martingale: $M_t = E[M_{t+s} \mid \mathcal{F}_t] \quad \forall s > 0$
  - "has increments with mean zero"
- a diffusion process with no drift is a martingale.

Thm: If M_t is a martingale on $\mathcal{F}_t^{\tilde{Z}}$ (Brownian motion), then there exists a process β_Z such that $dM_t = \beta_t dZ_t$

- we observe the path of the Brownian motion. If we take any process with zero mean increments, then it is a diffusion process with zero drift (and, by implication, continuous path)



- martingale, but not on a Brownian filtration

$M_t = 0$ on $(0,1)$

$M_t = 1$ on $[1,\infty)$ if $Z_1 \geq 0$

$M_t = -1$ on $[1,\infty)$ if $Z_1 < 0$

? is M_t a martingale on $\mathcal{F}_t^Z$?

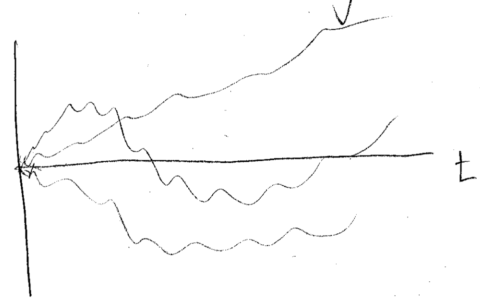
- No. If $\underset{\overset{\parallel}{0}}{M_{1/2}} = E[M_1 \mid \mathcal{F}_{1/2}^Z] = \begin{cases} 1 & Z_{1/2} \geq 0 \\ -1 & Z_{1/2} < 0 \end{cases}$

(?) How about on $\mathcal{F}_t^M$? Yes.

- Let $M_t = Z_t$. What is β_t? $dZ_t = \beta_t dZ_t \Rightarrow \beta_t = 1$
- $M_t = \Pr[Z_T \geq 0 \mid \mathcal{F}_t^Z]$. How to find β_t? Hint: Ito formula

  $= E[1\{Z_t \geq 0\} \mid \mathcal{F}_t^Z] \Rightarrow M_t$ is a martingale

Girsanov's Theorem



- change in measure that gives a process with different drift by choosing certain paths more often
  - ie changing the drift is equivalent to changing the probability distribution over the sample paths.

Suppose that under probability measure P, Z is a Brownian motion. Suppose that under Q, Z is not a Brownian motion: $dZ_t = \theta_t dt + d\tilde{Z}_t$

Girsanov's theorem gives us $\underbrace{\xi_t}_{\text{relative density}}$: $Q[\underbrace{A}_{\text{known at } t}] = E^P[1_A \cdot \xi_t]$

(rearranges the weights assigned to the statespace)

- $\xi_0 = 1$ and $d\xi_t = \theta_t \xi_t dZ_t$.
- observe: $E_0^P[\xi_t] = \xi_0 = 1$