Problem set I; due Sept. 28 in class

LATE PAPERS NOT ACCEPTED;

hand in what you have.

See the attached sheets for the "ground rules" for problem sets; in particular, what is meant by "good mathematical style."

(1.) Let $\{A_n\}$ be a family of subsets of a set X, indexed with the positive integers. Let

$$B_n = \bigcup_{k=n}^{\infty} A_k \quad \text{and} \quad C_n = \bigcap_{k=n}^{\infty} A_k.$$

define

$$\limsup A_n = \bigcap_{n=1}^{\infty} B_n$$

and $\liminf A_n = \bigcup_{n=1}^{\infty} C_n$.

(a) Show $x \in \limsup A_n \iff x \in A_k$ for infinitely many k.

(b) Show $x \in \liminf A_n \iff x \in A_k$ for all but finitely many k

[optional-extra credit] (c) Let $X = \mathbb{R}$ and let $A_k = \{ m/k \mid m \in \mathbb{Z} \}$

What are $\limsup A_k$ and $\liminf A_k$? Give answers only.

(2.) If $\mathcal{C}(\mathbb{R})$ is the collection of countable subsets of $\mathbb{R}$, show that $\mathcal{C}(\mathbb{R})$ and $\mathbb{R}$ have the same cardinality. [Hint: See the hint in Ex 8; §9. Recall that $\mathbb{R}$ and the interval $(0,1)$ of $\mathbb{R}$ have the same cardinality.]

(3.) [redacted] Determine the closures of the following subsets of the ordered square; give answers only

$$A = \{(1/n) \times 1/2 \mid n \in \mathbb{Z}_+\}$$

$$B = \{x \times 0 \mid 1/2 < x < 1\}$$

$$C = \{x \times 1/2 \mid x \text{ is rational and } 0 < x < 1\}.$$

(4.) [redacted] Consider the following topologies on $\mathbb{R}$:

$\mathcal{T}_1$ = the standard topology

$\mathcal{T}_2$ = the lower limit topology

$\mathcal{T}_3$ = the topology having as basis all open rays of the form $(a, +\infty)$ for $a \in \mathbb{R}$.

$\mathcal{T}_4$ = the topology having as basis all open intervals (a, b) and all one-point sets $\{c\}$ where c is irrational.

$\mathcal{T}_5$ = the topology having as basis all open intervals (a, b) and all sets of the form $(a, b) - \mathbb{Q}$, where $\mathbb{Q}$ is the set of rationals.

(a) Fill in each square with $\subset$ or $\supset$ or $=$ or N (not comparable)

$\mathcal{T}_1 \square \mathcal{T}_2$	$\mathcal{T}_2 \square \mathcal{T}_3$	$\mathcal{T}_3 \square \mathcal{T}_4$
$\mathcal{T}_1 \square \mathcal{T}_3$	$\mathcal{T}_2 \square \mathcal{T}_4$	$\mathcal{T}_3 \square \mathcal{T}_5$
$\mathcal{T}_1 \square \mathcal{T}_4$	$\mathcal{T}_2 \square \mathcal{T}_5$	$\mathcal{T}_4 \square \mathcal{T}_5$
$\mathcal{T}_1 \square \mathcal{T}_5$		

(b) Determine the closure of the interval $(\sqrt{2}, 3)$ in each of these topologies; give answers only.