Endogenous Quantile Regression

- $Y = D'\alpha(U) + X'\beta(U)$
- $D = \delta(X, Z, V)$
- $U \mid Z, X \sim \text{Uniform}(0,1)$
- $\tau \mapsto D'\alpha(\tau) + X'\beta(\tau)$ strictly increasing in τ.
- endogeneity due to relationship between U and V.
  - independence b/t U and Z is a crucial assumption.

Example: Demand and Supply

- $\ln Q_p = \alpha_0(U) + \alpha_1(U) \cdot \ln(p)$
- $\ln S_p = f(p, Z, V)$
- $P \in \{p : Q_p = S_p\}$
- $U \perp\!\!\!\perp Z \quad U \sim U[0,1]$

Equilibrium:

- $\ln Y = \alpha_0(U) + \alpha_1(U) \ln P$
- $P = \delta(Z, U, V, \text{"sunspots"})$
  $= \delta(Z, W)$
- $U \perp\!\!\!\perp Z$

- $(\underbrace{Y_0}_{\text{outcome if not trained}}, \underbrace{Y_1}_{\text{outcome if trained}})$

- $Y_d = q(d, U_d)$, $d \in \{0,1\}$, $U_d \sim U[0,1]$
- $q(d,\tau) = \alpha(\tau) + \delta(\tau) d$
- $D = \underset{d \in \{0,1\}}{\operatorname{argmax}} \; E[\text{Utility}(q(d, U_d), Z) | Z, V]$
- $(U_0, U_1) \perp\!\!\!\perp Z$

Then we have

- $Y = Y_D = q(D, U)$, $U = D \cdot U_1 + (1-D) U_0$
- $D = \delta(Z, V)$

When is $U \perp\!\!\!\perp Z$ satisfied?

- If $U = U_0 = U_1$
- $U_0 | Z, V = U_1 | Z, V$

How do we estimate such a model?

$$Pr[Y \le D'\alpha(\tau) + X'\beta(\tau) \mid X, Z] = \tau$$

since $\{Y \le D'\alpha(\tau) + X'\beta(\tau)\}$

$$\Leftrightarrow \{D'\alpha(U) + X'\beta(U) \le D'\alpha(\tau) + X'\beta(\tau)\}$$

$$\Leftrightarrow \{U \le \tau\}$$

Use GMM

Bayesian and Quasi-Bayesian Methods:

Suppose $E[(\tau - 1\{Y \le D'\theta\}) Z] = 0$

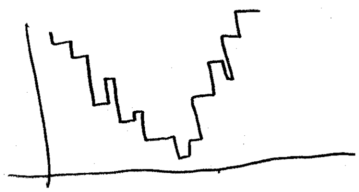
GMM: maximize $L_n(\theta) = -\frac{1}{2} g_n(\theta)' W(\theta) g_n(\theta)$ ◦ pseudo-likelihood function

with $g_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} (\tau - 1(Y_i \le D_i'\theta)) Z_i$

and $W(\theta) = \frac{1}{\tau(1-\tau)} \left(\frac{1}{n} \sum_{i=1}^{n} Z_i Z_i' \right)^{-1}$

◦ This is computationally difficult:

◦ objective function looks like this.



Define $p_n(\theta) = C \cdot \exp\{L_n(\theta)\}$

◦ $p_n(\theta)$ is the posterior

◦ C is s.t. $\int p_n(\theta)\, d\theta = 1$

If $L_n(\theta)$ is an actual likelihood function, this method reduces to the formal Bayesian framework

◦ an estimator: $\hat{\theta} = \int_{\Theta} \theta\, p_n(\theta)\, d\theta$

◦ defined by integration.

◦ This integral will be computed using MCMC numerical integration approach.

◦ creates a sample $S = (\theta^{(1)}, \ldots, \theta^{(k)})$ with marginal distribution $C \cdot \exp\{L_n(\theta)\}$

Criterion function $L_n(\theta)$

If $\frac{L_n}{n} \to M$ where M is optimized at θ_0, then $\hat{\theta}_n \xrightarrow{P} \theta_0$

$L_n(\theta)$ is not a likelihood. Let $\pi(\theta)$ be the prior

Then $$p_n(\theta) = \frac{e^{L_n(\theta)}\pi(\theta)}{\int_\Theta e^{L_n(\theta)}\pi(\theta)d\theta}$$

$$\Rightarrow \int_\Theta p_n(\theta)d\theta = 1.$$

We can just write $p_n(\theta) \propto e^{L_n(\theta)}\pi(\theta)$

Then we have that $\hat{\theta} = \int_\Theta \theta\, p_n(\theta)d\theta$

$\overline{\overline{QBE}}$ is defined by (under belief p_n)

$$\hat{\theta} = \underset{d \in \Theta}{\operatorname{argmin}} \left[E_{p_n}\left[\rho_n(d-\theta)\right]\right]$$

$$= \underset{d \in \Theta}{\operatorname{argmin}} \left[\int_\Theta \rho_n(d-\theta)\frac{e^{L_n(\theta)}\pi(\theta)}{\int_\Theta e^{L_n(\theta)}\pi(\theta)d\theta} d\theta\right]$$

where $\rho_n(u)$ is the loss function