Last time, we discussed asymptotic results for GMM

$$\sqrt{T}(\hat\beta - \beta_0) \xrightarrow{d} N\left(0, (G'AG)^{-1} G' A \Omega A G (G'AG)^{-1}\right)$$

$$G = E\left[\frac{\partial g_i(\beta_0)}{\partial \beta}\right], \quad \Omega = E[g_i(\beta_0) g_i(\beta_0)']$$

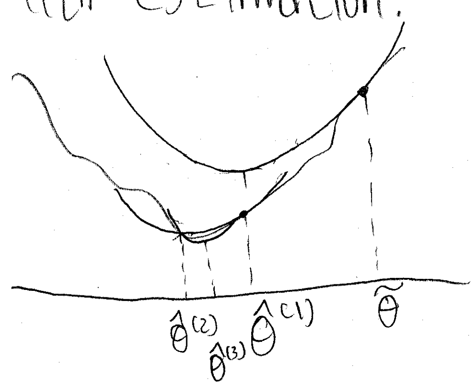
Two-step estimators:

Suppose $E[m(z_i, \beta, \gamma_0)] = 0$. Using some preliminary moment conditions, $E[h(z_i, \gamma_0)] = 0$, we estimate a first step $\hat\gamma_{GMM} = \operatorname{argmin} E_n[h]' E_n[h]$. Under regularity conditions, $\hat\gamma_{GMM} \to \gamma_0$. We can then use this to estimate $\hat\beta_{GMM}$. This will be asymptotically equivalent to estimating $(\hat\beta_{GMM}, \hat\gamma_{GMM})$ on $g(z, \theta) = \begin{bmatrix} m(z, \beta, \gamma) \\ h(z, \gamma) \end{bmatrix}$.

One-step estimation:

Important for numerical estimation.

Newton-Raphson:



Minimize $Q_n(\theta) \approx Q_n(\hat\theta) + \nabla Q_n(\hat\theta)(\theta - \hat\theta) + \frac{1}{2}(\theta - \hat\theta)\nabla^2 Q_n(\hat\theta)(\theta - \hat\theta)$

FOCs give $\nabla Q_n(\hat\theta) + \nabla^2 Q_n(\hat\theta)(\bar\theta - \hat\theta) = 0$

$$\Rightarrow \bar{\theta} = \hat{\theta}^{(1)} = \hat{\theta} - [\nabla^2 Q_n(\hat{\theta})]^{-1} \nabla Q_n(\hat{\theta})$$

$$\Rightarrow \hat{\theta}^{(2)} = \hat{\theta}^{(1)} - [\nabla^2 Q_n(\hat{\theta}^{(1)})]^{-1} \nabla Q_n(\hat{\theta}^{(1)})$$

$$\vdots$$

$$\Rightarrow \hat{\theta}^{(n)} = \hat{\theta}^{(n-1)} - [\nabla^2 Q_n(\hat{\theta}^{(n-1)})]^{-1} \nabla Q_n(\hat{\theta}^{(n-1)})$$

Suppose $\hat{\theta} - \hat{\theta}^{(\infty)} = O_p(\frac{1}{\sqrt{n}})$, where $\hat{\theta}^{(\infty)} = \lim_{n\to\infty} \hat{\theta}^{(n)}$

Then $\sqrt{n}(\hat{\theta}^{(1)} - \hat{\theta}^{(\infty)}) \xrightarrow{P} 0$

- However, we can improve the rate of convergence by iterating $\hat{\theta}^{(n)}$.

$$\sqrt{n}(\hat{\theta}^{(1)} - \theta_0) = \sqrt{n}(\hat{\theta} - \theta_0) - \nabla^2 Q_n(\hat{\theta})^{-1} \sqrt{n}\, Q_n(\hat{\theta})$$

$$= \sqrt{n}(\hat{\theta} - \theta_0) - \nabla^2 Q_n(\hat{\theta})^{-1}[\sqrt{n}\nabla Q_n(\theta_0) + \sqrt{n}(\hat{\theta} - \theta_0)\nabla^2 Q_n(\theta^*)]$$

$$= (I - \nabla^2 Q_n(\hat{\theta})^{-1} \nabla^2 Q_n(\theta^*))\sqrt{n}(\hat{\theta} - \theta_0) - \nabla^2 Q_n(\hat{\theta})^{-1}\sqrt{n}\nabla Q_n(\theta_0)$$

$$= o_p(1) + \sqrt{n}(\hat{\theta}^{\infty} - \theta_0)$$

- asymptotic variance is minimized by choosing $\hat{A}$ s.t. $\hat{A} \xrightarrow{P} \Omega^{-1}$.
- MLE is efficient estimator in the class of GMM estimators
  - follows by the standard argument.
  - alternatively, Gauss-Markov works

MLE as the optimal GMM follows under regularity conditions since MLE achieves asymptotic CRLB.

Barry-Levinsohn-Pakes - extend multinomial choice

- j - indexes goods $j = 1, \dots, J$
- m - markets $m = 1, \dots, M$ (number of obs.)
- Let θ be unknown parameters. Want to estimate θ_0
- s_{jm} - share of commodity j in mkt m, $\tilde{s}_m = (s_{1m}, \dots, s_{Jm})$
- p_{jm} - price of commodity j in mkt m $\tilde{p}_m = (p_{1m}, \dots, p_{Jm})$
- x_{jm} - observable characteristics, $\tilde{x}_m = (x_{1m}, \dots, x_{Jm})$
- ε_{jm} - unobservable characteristics that are correlated with prices. (source of endogeneity)
- $s_{jm}(\hat{\varepsilon}_m, \hat{p}_m, \tilde{x}_m; \theta)$ is specified by economic model
  - e.g. $s_{jm}(\hat{\varepsilon}_m, \tilde{p}_m, \tilde{x}_m; \theta) = \dfrac{\exp\{p_{jm}\beta_j + x_{jm}'\gamma_j + \varepsilon_{jm}\}}{\sum_{k=1}^{J} \exp\{\beta_k p_{km} + x_{km}'\gamma_k + \varepsilon_{km}\}}$

    - if prices were exogenous, we would just do logit.

  - can solve for all the ε_{jm}'s
    - $\varepsilon_{jm} = \rho_j(\tilde{x}_m, \tilde{p}_m, \tilde{s}_m, \theta_0)$

- assume z_{jm} vector of instruments orthogonal to ε_{jm}. That is, $E[z_{jm}\varepsilon_{jm}]=0$

Choice of instruments

- z_{jm} is a function of $\tilde{X}_m$
    - assumes that observables are uncorrelated with unobservables.
    - This derives identification just through functional form assumptions.
- z_{jm} are functions of $p_{jm'}$ for $m' \neq m$.