Hint: 1 and 2 - consistency fails (identification does not hold)

$\sqrt{N}(\hat{\theta}-\theta_0) \xrightarrow{d} N(0, A_0^{-1}B_0A_0^{-1})$ for extremum estimators

$A_0 = E[H(w,\theta_0)]$

$B_0 \equiv E[s(w,\theta_0)s(w,\theta_0)'] = Var[s(w,\theta_0)]$

Thus, $\text{Avar}\,\hat{\theta} = \dfrac{A_0^{-1}B_0A_0^{-1}}{N}$

$\hat{A} = \frac{1}{N}\sum_{i=1}^{N} H(w_i,\hat{\theta}) \to A_0$

$\hat{B} = \frac{1}{N}\sum_{i=1}^{N} s(w_i,\hat{\theta})s(w_i,\hat{\theta})' \to B_0$

Testing

Wald: $H_0: \underset{Q\times 1}{c(\theta_0)} = 0$ In large samples, $c(\hat{\theta}) \approx 0$

Let $C(\theta) = \underset{Q\times K}{\dfrac{\partial c(\theta)}{\partial \theta'}}$

Delta method gives us: $\sqrt{N}(\hat{\theta}-\theta_0) \xrightarrow{d} N(0,V)$

$\Rightarrow \sqrt{N}(c(\hat{\theta}) - \underbrace{c(\theta_0)}_{=0 \text{ under } H_0}) \xrightarrow{d} N(0, \underset{Q\times K}{C}\underset{K\times K}{V}\underset{K\times Q}{C'})$

$\Rightarrow \sqrt{N}\,c(\hat{\theta}) \xrightarrow{d} N(0, CVC')$

$\Rightarrow \sqrt{N}(\hat{C}\hat{V}\hat{C}')^{-1/2}c(\hat{\theta}) \xrightarrow{d} N(0, I_Q)$

$\Rightarrow N\,c(\hat{\theta})'(\hat{C}\hat{V}\hat{C}')^{-1}c(\hat{\theta}) \xrightarrow{d} \chi^2(Q)$

Drawbacks:
- Need $\theta_0 \in \text{int}(\Theta)$
- Not invariant to how restriction is written.

Advantage: Need only compute $\hat{\theta}$ unrestricted.

Lagrange Multiplier Test (Score test)

Score at restricted estimator should be approximately zero

$$\hat{S}^R = \sqrt{N}\ \frac{1}{N}\sum_{i=1}^{N} s_i(\hat{\theta}^R)$$

It turns out that

$$LM \equiv \hat{S}^{R\prime} A_0^{-1} C_0' (C_0 A_0^{-1} B_0 A_0^{-1} C_0')^{-1} C_0 A_0^{-1} \hat{S}^R \xrightarrow{d} \chi^2(Q)$$

$\hat{LM}$ involves evaluating everything at $\hat{\theta}^R$.

Likelihood Ratio

Change in objective function under null from $\hat{\theta}^{UR}$ to $\hat{\theta}^R$ should be small

$$QLR = 2(Q(\hat{\theta}^{UR}) - Q(\hat{\theta}^R)) \xrightarrow{d} \chi^2(Q) \text{ under } H_0$$

Disadvantages:
- Need GIME (generalized information matrix equality) to hold.
- Have to compute $\hat{\theta}^{UR}$ and $\hat{\theta}^R$.

- $T(y_i, \lambda_0) = X_i' \delta_0 + u_i$, $E[u_i | X_i] = 0$

i) Want to find Z such that $E[Zu] = 0$

- By $E[u_i|X_i] = 0$ $E[h(X)u] = 0$ for any function h

$$\Rightarrow E\left[h(X)\left(\frac{y^\lambda - 1}{\lambda} - X'\delta_0\right)\right] = 0$$

ii) For $\beta = (\lambda, \delta')'$, find a nice function $g_i(\beta)$ s.t. $E[g_i(\beta_0)] = 0$

- Let $h(X_i) = \begin{bmatrix} X_{i1} \\ \vdots \\ X_{iK} \\ X_{i1}^2 \\ \vdots \\ X_{iK}^2 \end{bmatrix} \Rightarrow g_i(\beta) = h(X_i)\left(\frac{y_i^{\lambda}-1}{\lambda} - X_i'\delta \right)$

- $\underset{2K\times 1}{E[g_i(\beta_0)]} = 0$

$\Rightarrow$ Let $\bar{g}(\beta) = \frac{1}{N}\sum_{i=1}^{N} g_i(\beta)$. Then we want to

$\min_{\beta} \bar{g}(\beta)'\hat{A}^{-1}\bar{g}(\beta)$. This will give us $\hat{\beta}_{GMM}$

iii) First, let us find <u>any</u> GMM estimator. Let $\hat{A}^{-1} = I$. Then

$$\begin{aligned}\hat{\beta}^{I}_{GMM} &= \operatorname{argmin} \bar{g}(\beta)'\bar{g}(\beta) \\ &= \operatorname{argmin} \sum_{l=1}^{2K} \left(\bar{g}^{l}(\beta)\right)^2 \\ &= \operatorname{argmin} \sum_{l=1}^{2K} \left(\sum_{i=1}^{N} g_i^{l}(\beta)\right)^2\end{aligned}$$

This will be inefficient.

Define $\hat{\Omega} = \frac{1}{N}\sum_{i=1}^{N} g_i(\hat{\beta}^{I}_{GMM})\, g_i(\hat{\beta}^{I}_{GMM})'$

$\rightarrow \text{Avar}(g_i(\hat{\beta}^{I}_{GMM}))$

Then let $\hat{\beta}^{2}_{GMM} = \operatorname{argmin} \bar{g}(\beta)'\hat{\Omega}^{-1}\bar{g}(\beta)$