14.383 Fall 2005 Final Exam

- General SEM: SF: $\underset{(T\times M)}{Y}\underset{(M\times M)}{B} + \underset{(T\times K)}{Z}\underset{(K\times M)}{\Gamma} = \underset{(T\times M)}{U}$ RF: $Y = Z\Pi + V$

  just one point in t: $y_t' B + z_t'\Gamma = u_t'$, $Var(u_t) = \Sigma$. $y_t' = z_t'\Pi + v_t'$, $Var(v_t) = \Omega$

  just one equation: $\underset{(T\times 1)}{y_j} = \underset{(T\times k_j)}{X_j}\underset{(k_j\times 1)}{\delta_j} + \underset{(T\times 1)}{u_j}$, $X_j \equiv [\underset{(T\times r_j)}{Y_j} \ \ \underset{(T\times s_j)}{Z_j}]$

  $\delta_j \equiv [\underset{(r_j\times 1)}{B_{(j)}^{u}}{}' \ \ \underset{(s_j\times 1)}{\Gamma_{(j)}^{u}}{}']$

  in stacked form: $\underset{(MT\times 1)}{\bar{y}} = \underset{(MT\times \sum k_j)}{\bar{X}}\underset{(\sum k_j\times 1)}{\bar{\delta}} + \underset{(MT\times 1)}{\bar{u}}$ $\underset{(MT\times 1)}{\bar{y}} = \underset{(MT\times MK)}{\bar{\bar{Z}}}\underset{(MK\times 1)}{\bar{\Pi}} + \underset{(MT\times 1)}{\bar{v}}$.

- Identification of equation "j" with linear restrictions $\underset{(g_j\times(M+K))}{\Phi_j}\cdot\underset{((M+K)\times 1)}{\begin{bmatrix} B_{(j)} \\ \Gamma_{(j)} \end{bmatrix}} = \underset{(1\times g_j)}{\phi_j}$:
  - order condition: $g_j \geq M$ (necessary)
  - rank condition: $\text{rank}\begin{bmatrix} \Pi & I_K \\ \hline \multicolumn{2}{c}{\Phi_j} \end{bmatrix} = M+K \iff \text{rank}\left(\Phi_j\cdot\begin{bmatrix} B \\ \Gamma\end{bmatrix}\right) = M$ (necessary and sufficient).

- Notation:
  $\frac{1}{T}Z'Z \xrightarrow{p} Q_{ZZ}$; $\frac{1}{T}Z'X_j = \frac{1}{T}Z'[Z\pi_j + v_j \ \ Z S_j] \xrightarrow{p} Q_{ZZ}[\pi_j \ \ S_j] \equiv Q_{ZZ}D_j$; $\frac{1}{T}Z'u_j u_j' Z \xrightarrow{p} V_Z$; $V \equiv E[u_j u_j']$

- LI estimators: $\hat{A} \xrightarrow{p} A$, a full rank $(K\times k_j)$ matrix.

- IV: $\hat{\delta}_{j,IV} = (W'X_j)^{-1}W'y_j$, where $\underset{(T\times k_j)}{W} = \underset{(T\times K)}{Z}\underset{(K\times k_j)}{\hat{A}}$ so: $\hat{\delta}_{j,IV} = (\hat{A}'Z'X_j)^{-1}\hat{A}'Z'y_j$

  $\Rightarrow \sqrt{T}(\hat{\delta}_{j,IV} - \delta_j) \xrightarrow{d} N(0, [A'Q_{ZZ}D_j]^{-1}A'V_Z A[D_j'Q_{ZZ}A]^{-1})$.

- 2SLS: $\hat{\delta}_{j,2SLS} = [X_j'Z(Z'Z)^{-1}Z'X_j]^{-1}X_j'Z(Z'Z)^{-1}Z'y_j = (X_j'P_Z X_j)^{-1}X_j'P_Z y_j = (\hat{X}_j'\hat{X}_j)^{-1}\hat{X}_j'y_j =$

  $= \underset{\delta_j}{\text{argmin}}\ (y_j - X_j\delta_j)'Z(Z'Z)^{-1}Z'(y_j - X_j\delta_j)$, Here, $W = Z\underbrace{(Z'Z)^{-1}Z'X_j}_{\hat{A}} = P_Z X_j = \hat{X}_j$.

  $\Rightarrow \sqrt{T}(\hat{\delta}_{j,2SLS} - \delta_j) \xrightarrow{d} N(0, [D_j'Q_{ZZ}D_j]^{-1}D_j'V_Z D_j[D_j'Q_{ZZ}D_j]^{-1})$.

- 2SLS equation by equation:

  $\hat{\bar{\delta}}_{2SLS} = [\bar{X}'(I_M\otimes P_Z)\bar{X}]^{-1}\bar{X}'(I_M\otimes P_Z)\bar{y}$, so $\bar{W} = (I_M\otimes Z)\bar{A} = (I_M\otimes Z)\cancel{(I_M\otimes I_T)}(I_M\otimes(Z'Z)^{-1}Z')\bar{X}$.

- OIV: $\hat{\delta}_{j,OIV} = (X_j'Z\hat{V}_Z^{-1}Z'X_j)^{-1}X_j'Z\hat{V}_Z^{-1}Z'y_j$, so $W = Z\hat{A} = Z\hat{V}_Z^{-1}Z'X_j$

  $= \underset{\delta_j}{\text{argmin}}\ (y_j - X_j\delta_j)'Z\hat{V}_Z^{-1}Z'(y_j - X_j\delta_j)$

  $\Rightarrow \sqrt{T}(\hat{\delta}_{j,OIV} - \delta_j) \xrightarrow{d} N(0, [D_j'Q_{ZZ}V_Z^{-1}Q_{ZZ}D_j]^{-1})$.

  If $V_Z = \sigma^2 Q_{ZZ} \Rightarrow AsyVar(\hat{\delta}_{OIV}) = AsyVar(\hat{\delta}_{2SLS})$.

  If $K = k_j \Rightarrow \hat{A}$, $Z'X$ invertible $\Rightarrow AsyVar(\hat{\delta}_{OIV}) = AsyVar(\hat{\delta}_{2SLS}) = AsyVar(\hat{\delta}_{IV})$.

  2SLS finite-sample bi[as]

  $bias(\hat{\delta}_{j,2SLS}) = \frac{K\sigma_{u_j v}}{T R_{FS}^2 Var(Y_j)} = \frac{K\sigma_{u_j v}}{T\cdot\pi_j'Q_{ZZ}\pi_j + K\sigma_v^2}$

- LIML: $\hat{\delta}_{j,LIML} = (\hat{D}_j'Z'X_j)^{-1}\hat{D}_j'Z'y_j$

  $\Rightarrow \sqrt{T}(\hat{\delta}_{j,LIML} - \delta_j) \xrightarrow{d} N(0, [D_j'Q_{ZZ}D_j]^{-1}D_j'Q_{ZZ}V_Z Q_{ZZ}D_j[D_j'Q_{ZZ}D_j]^{-1})$.

  [LIML is derived as FIML on $\left\{\begin{array}{l} y_j = X_j\delta_j + u_j \\ Y_j = Z\pi_j + v_j \end{array}\right\}$].

- GMM: $\hat{\delta}_{j,GMM} = \underset{\delta_j}{\text{argmin}}\ \hat{g}(\delta)'\hat{A}\hat{g}(\delta)$, where $\left\{\begin{array}{l} \hat{g}(\delta) = \frac{1}{T}\sum_{t=1}^{T} g_t(\delta) \\ \hat{A} \xrightarrow{p} \tilde{A} \\ \sqrt{T}\hat{g}(\delta) \xrightarrow{d} N(0, V_g) \end{array}\right.$, $E[g_t(\delta)] = 0$ moment condition

  $\to \sqrt{T}(\hat{\delta}_{j,GMM} - \delta_j) \xrightarrow{d} N\Big(0, \Big[\underbrace{\Big(\frac{\partial\hat{g}(\delta)}{\partial\delta}\Big)'}_{G'}\tilde{A}\underbrace{\Big(\frac{\partial\hat{g}(\delta)}{\partial\delta}\Big)}_{G}\Big]^{-1}G'\tilde{A}V_g\tilde{A}G[G'AG]^{-1}\Big)$.

  (optimal GMM sets $\tilde{A} = V_g^{-1}$)

- NL2SLS:

  $\hat{\delta}_{j,NL2SLS} = \underset{\delta_j}{\text{argmin}}\ [\underbrace{y_j - h_j(X_j,\delta_j)}_{f_j(y_j,X_j,\delta_j)}]'\tilde{Z}_j(\tilde{Z}_j'\tilde{Z}_j)^{-1}\tilde{Z}_j'[y_j - h_j(X_j,\delta_j)]$

- NLOIV: $\hat{\delta}_{j,NLOIV} = \underset{\delta_j}{\text{argmin}}\ [y_j - h_j(X_j,\delta_j)]'\tilde{Z}_j V_{\tilde{Z}_j}^{-1}\tilde{Z}_j'[y_j - h_j(X_j,\delta_j)]$

  where: $\tilde{Z}_j$ contains Z but also non-linear functions of Z.

· FI estimators:

– 3SLS: $\hat{\delta}_{3SLS} = [\bar{X}'(\hat{\Sigma}^{-1}\otimes P_Z)\bar{X}]^{-1}\bar{X}'(\hat{\Sigma}^{-1}\otimes P_Z)\bar{y}$, so: $\bar{W} = (I_M\otimes Z)\bar{A} = (I_M\otimes Z)(\hat{\Sigma}^{-1}\otimes I_T)(I_M\otimes(Z'Z)^{-1}Z')\bar{X}$

$\Rightarrow \sqrt{T}(\hat{\delta}_{3SLS}-\bar{\delta}) \xrightarrow{d} N(0,[\bar{D}'(\Sigma^{-1}\otimes Q_{ZZ})\bar{D}]^{-1})$, where: $\bar{D} = \begin{bmatrix} D_1 & & 0 \\ & D_j & \\ 0 & & D_M\end{bmatrix}$ $(=\operatorname{diag}(D_j))$

If $K=K_j$ $\forall j$, then $\hat{\delta}_{3SLS} = \hat{\delta}_{2SLS}$.

– FIML: $\hat{\delta}_{FIML} = [\hat{\bar{D}}'(\hat{\Sigma}_{ML}^{-1}\otimes Z')\bar{X}]^{-1}\hat{\bar{D}}'(\hat{\Sigma}_{ML}^{-1}\otimes Z')\bar{y}$, so: $\bar{W} = (I_M\otimes Z)\bar{A} = (I_M\otimes Z)(\hat{\Sigma}_{ML}^{-1}\otimes I_T)\hat{\bar{D}}$

$\Rightarrow \sqrt{T}(\hat{\delta}_{FIML}-\bar{\delta}) \xrightarrow{d} N(0,[\bar{D}'(\Sigma^{-1}\otimes Q_{ZZ})\bar{D}]^{-1})$ and $\hat{\bar{D}} = \operatorname{diag}(\hat{D}_j) = \operatorname{diag}([(\hat{\Gamma}_{FIML}\hat{B}_{FIML})_j \;\; S_j])$.

[FIML solves $\max_{B,\Gamma,\Sigma^{-1}} \mathcal{L}(B,\Gamma,\Sigma^{-1}) = ct. + \frac{T}{2}\log|\det\Sigma^{-1}| - \frac{1}{2}tr[\Sigma^{-1}U'U] + T\cdot\log|\det B'|$

↪ FOC: $\{D_j'Z'U\Sigma_{(j)}^{-1} = 0\}_{j=\{1,\dots,M\}} \Leftrightarrow \bar{D}\cdot Z'U\Sigma^{-1} = 0$.]

– NL3SLS (BHHH): $\hat{\delta}_{NL3SLS} = \arg\min_{\bar{\delta}} \underbrace{[\bar{y}-h(\bar{X},\bar{\delta})]}_{f(\bar{y},\bar{X},\bar{\delta})}{}'(\hat{\Sigma}^{-1}\otimes I_T)\underbrace{\operatorname{diag}(\tilde{Z}_j)}_{\bar{H}}[\bar{H}(\hat{\Sigma}^{-1}\otimes I_T)\bar{H}]^{-1}\bar{H}(\hat{\Sigma}^{-1}\otimes I_T)[\bar{y}-h(\bar{X},\bar{\delta})]$

· Hausman test: This test requires for $\underset{(K_j\times1)}{\hat{\delta}_0}$, $\underset{(K_j\times1)}{\hat{\delta}_1}$

1) $\hat{\delta}_0,\hat{\delta}_1$ have same plim under H_0
2) $\hat{\delta}_0,\hat{\delta}_1$ have different plims under H_1.
3) One estimator be efficient under H_0. W.l.o.g., assume $\hat{\delta}_1$ efficient under H_0:

$$H = T(\hat{\delta}_0-\hat{\delta}_1)'[\widehat{AsyVar}(\hat{\delta}_0)-\widehat{AsyVar}(\hat{\delta}_1)]^{-1}(\hat{\delta}_0-\hat{\delta}_1) \xrightarrow{d} \chi^2_{(K_j)}$$

· "Omnibus" test – LI:

$W = T\cdot\frac{\hat{u}'Z(Z'Z)^{-1}Z'\hat{u}}{\hat{u}'\hat{u}} \xrightarrow{d} \chi^2_{(K-K_j)} \Leftrightarrow T\cdot R^2 \xrightarrow{d} \chi^2_{(K-K_j)}$ where R^2 corresponds to the regression of $\hat{u}$ on Z.

or: $W = \frac{1}{T}\hat{u}'Z\hat{V}^{-1}Z'\hat{u} \xrightarrow{d} \chi^2_{(K-K_j)}$ if we have non-sphericality. (K: "dim(Z)", K_j: "dim(X_j)")

· "Omnibus" test – FI:

$$W = \hat{u}'(\hat{\Sigma}^{-1}\otimes P_Z)\hat{u} \xrightarrow{d} \chi^2_{(\sum_{j=1}^{M}(K-K_j))}$$

$$LR = 2[\log\mathcal{L}(\hat{\Pi}_u,\hat{\Omega}_u) - \log\mathcal{L}(\hat{\Pi}_R,\hat{\Omega}_R)] \xrightarrow{d} \chi^2_{(\sum_{j=1}^{M}(K-K_j))}$$
$(= T\cdot\log(\det\hat{\Omega}_R/\det\hat{\Omega}_u))$

· Testing on the RF:

H_0: $h(\pi) = 0_{(p\times1)}$
H_1: $h(\pi) \neq 0_{(p\times1)}$

$$W = T\cdot h(\hat{\pi})'[\widehat{AsyVar}(h(\hat{\pi}))]^{-1}h(\hat{\pi}) \xrightarrow{d} \chi^2_p$$

If $h(\pi)$ is not linear, then do:

$W = h(\hat{\pi}_u)'[H_u\hat{\Psi}_u^{-1}H_u']^{-1}h(\hat{\pi}_u) \xrightarrow{d} \chi^2_p$, where: $H_u \equiv \frac{\partial h(\pi)}{\partial\pi}\Big|_{\hat{\pi}_u}$; $\hat{\Psi}^{-1} = \bar{Z}'(\hat{\Omega}_u^{-1}\otimes I_T)\bar{Z}$.

or

$LM = (\hat{\pi}_u-\hat{\pi}_R)'\hat{\Psi}_u(\hat{\pi}_u-\hat{\pi}_R) \xrightarrow{d} \chi^2_p$.

and remember that the invariance theorem applies to ML estimators, so you can use:

$$\hat{\Pi}_R = -\hat{\Gamma}_{FIML}\cdot\hat{B}_{FIML}^{-1} \quad , \quad \hat{\Omega}_R = (\hat{B}_{FIML}^{-1})'\cdot\hat{\Sigma}_{FIML}\cdot\hat{B}_{FIML}^{-1}.$$

Cheat Sheet (cont...)

Weak Instruments

- $y_1 = \beta y_2 + \varepsilon_1$ $\quad$ $\dim(\pi_2) = K$ $\quad$ $\begin{pmatrix} \varepsilon_{1i} \\ v_{2i} \end{pmatrix} \sim N(0, \Sigma)$

  $y_2 = Z\pi_2 + v_2$

$$\Sigma = \begin{pmatrix} \sigma_{\varepsilon\varepsilon} & \sigma_{\varepsilon v} \\ \sigma_{\varepsilon v} & \sigma_{vv} \end{pmatrix} = \begin{pmatrix} 1 & e \\ e & 1 \end{pmatrix}$$

$\mathrm{Var}(y_2) = 1/(1-R^2)$

2SLS bias $\quad E(\beta_{2SLS} - \beta) \simeq \dfrac{K\sigma_{\varepsilon v}}{1/n(\pi' Z' Z\pi)} \simeq \dfrac{K\sigma_{\varepsilon v}}{nR^2 \mathrm{Var}(y_2)} = \dfrac{Ke(1-R^2)}{nR^2}$

Let $\Theta = \pi' Z' Z\pi / n$

$$E(\hat{\sigma}^2_{2SLS}) \simeq \sigma_{\varepsilon\varepsilon} - \frac{2}{n}\frac{(K-2)\sigma^2_{\varepsilon v}}{\Theta} - \frac{1}{n}\sigma^2_{\varepsilon v} + \frac{1}{n}\frac{\sigma_{\varepsilon\varepsilon}\sigma_{vv}}{\Theta}$$

HH Test $\quad \sqrt{n}\left(\beta_{2SLS} - \dfrac{1}{c_{2SLS}} - \hat{B}\right) \to N(0, V)$

- MSE : JK2SLS, Nagar $\quad \dfrac{\sigma_{\varepsilon\varepsilon}}{\Theta} + \dfrac{K}{n}\dfrac{\sigma_{\varepsilon\varepsilon}\sigma_{vv} + \sigma_{\varepsilon v}}{\Theta^2} = \dfrac{(1-R^2)}{R^2} + \dfrac{K(1+e^2)}{n}\left(\dfrac{1-R^2}{R^2}\right)^2$

MSE : Fuller $\quad \dfrac{\sigma_{\varepsilon\varepsilon}}{\Theta} + \dfrac{K}{n}\dfrac{\sigma_{\varepsilon\varepsilon}\sigma_{vv} - \sigma_{\varepsilon v}}{\Theta^2} = \dfrac{(1-R^2)}{R^2} + \dfrac{K(1-e^2)}{n}\left(\dfrac{1-R^2}{R^2}\right)^2$

Kleibergen statistic for null hypothesis of β_0

$$K(\beta_0) = \frac{\varepsilon_0' P_{Y_2(\beta_0)} \varepsilon_0}{\frac{1}{n-K}\varepsilon_0' Q_Z \varepsilon_0}$$

where $\varepsilon_0 = y_1 - \beta_0 y_2$

$Y_2(\beta_0) = Z\pi_2(\beta_0)$

$$\pi_2(\beta_0) = (Z'Z)^{-1} Z' [Y_2 - \varepsilon_0 \sigma_{\varepsilon v}(\beta_0) / \sigma_{\varepsilon\varepsilon}(\beta_0)]$$

- Invalid instruments $\quad \mathrm{plim}(\beta_{2SLS} - \beta) \simeq \sigma_{W'\varepsilon}\dfrac{1-R^2}{R^2}, \quad W = Z\pi_2$