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HO: 2.4

Simultaneous Equations

Let's imagine that observations are made over time so that the i^{th} observation is collected at time i:.

At each i, we observe the row vector

$$\underset{1\times m}{(Y_i}, \underset{1\times K}{Z_i)}$$

and we imagine that they are related by a system of m equations:

$$\underset{1\times m}{Y_i}\,\underset{m\times m}{B} + \underset{1\times K}{Z_i}\,\underset{K\times m}{\Gamma} = \underset{1\times m}{U_i}, \quad i=1,\dots,n \qquad E(U_i)=0, \quad E(U_i'U_i)=\Sigma$$

This is the <u>Structural Form</u>.

All simultaneous equations models can be written in this way.

<u>Example 1</u>:

$$y_{1t} = \beta_1 y_{2t} + \gamma_1 z_{2t} + u_{1t}$$
$$y_{2t} = \beta_2 y_{1t} + \gamma_2 z_{1t} + u_{2t}$$

$$Y_t = (y_{1t} \ \ y_{2t}) \qquad Z_t = (z_{1t} \ \ z_{2t}) \qquad U_t = (u_{1t} \ \ u_{2t})$$

$$B = \begin{pmatrix} 1 & -\beta_2 \\ -\beta_1 & 1 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 0 & -\gamma_2 \\ -\gamma_1 & 0 \end{pmatrix}$$

Y_i and Z_i represent the m endogenous and K exogenous variables, respectively. Z_i, U_i are independent.

Let $A := \begin{bmatrix} B \\ \Gamma \end{bmatrix}$. A collects the structural form parameters. Think of $\begin{bmatrix} B \\ \Gamma \end{bmatrix}$ as m horizontally stacked column vectors.

$$\begin{bmatrix} B \\ \Gamma \end{bmatrix} = \left[\begin{matrix} B_1 \\ \Gamma_1 \end{matrix} \middle| \cdots \middle| \begin{matrix} B_m \\ \Gamma_m \end{matrix} \right]$$

The j^{th} column of $\begin{bmatrix} B \\ \Gamma \end{bmatrix}$ contains the coefficients of the j^{th} equation.

Stacking the observations vertically, we have

$$\begin{matrix} Y_1 B + Z_1 \Gamma = U_1 \\ \vdots \\ Y_n B + Z_n \Gamma = U_n \end{matrix} \quad \Rightarrow \quad \underset{n\times m}{Y} \underset{m\times m}{B} + \underset{n\times K}{Z} \underset{K\times m}{\Gamma} = \underset{n\times m}{U}$$

The i^{th} row of $[Y, Z]$ is the observation $[Y_i, Z_i]$.

<u>Example 2</u>: From example 1, for $t = 1, \ldots, T$, we can group Y_t, Z_t so that

$\underset{T\times 2}{Y} \underset{2\times 2}{B} + \underset{T\times 2}{Z} \underset{2\times 2}{\Gamma} = \underset{T\times 2}{U}$. The t^{th} row of $[Y, Z]$ is observation $[Y_t, Z_t] = [y_{1t}\ y_{2t},\ z_{1t}\ z_{2t}]$

The 1st column of $\begin{bmatrix} B \\ \Gamma \end{bmatrix}$ is $\begin{pmatrix} 1 \\ -\beta_1 \\ 0 \\ \gamma_1 \end{pmatrix}$, the coefficients for Eq. (1). ∎

We are now ready to give the <u>Reduced Form</u>

$$YB + Z\Gamma = U$$

$$Y = -Z\Gamma B^{-1} + UB^{-1}$$

$$\underset{n\times m}{Y} = \underset{n\times k}{Z}\,\underset{k\times m}{\Pi} + \underset{n\times m}{V}, \qquad \Pi = -\Gamma B^{-1}$$

$$V = UB^{-1}$$

Think back to the reduced form for y_t, p_t given in the introduction. There, y_t and p_t were expressed in terms of the 1 exogenous varia I_t. In this general framework, there are K exogenous variables. The reduced form expresses each of the m endogenous variable in terms of the K exogenous variables.

Look at $\underset{K\times m}{\Pi}$. Π has on each column j the reduced form for the j^{th} equation

Considering only the observation at time i, we look at the i^{th} row of the reduced form:

$$\underset{1\times m}{Y_i} = \underset{1\times k}{Z_i}\,\underset{k\times m}{\Pi} + \underset{1\times m}{V_i}, \qquad \underset{1\times m}{V_i} = \underset{1\times m}{U_i}\,B^{-1}$$

The covariance matrix for the reduced form errors is $E(V_i'V_i) = E[(U_iB^{-1})'(V_iB^{-1})]$

$$= B^{-1'} E(U_i'U_i) B^{-1}$$

$$= B^{-1'} \Sigma B^{-1} = \Omega$$

<u>Example 3</u>: Continuing examples 1, 2, we have

$$\underset{T\times 2}{Y} = \underset{T\times 2}{Z}\,\underset{2\times 2}{\Pi} + \underset{T\times 2}{V}, \quad \Pi = -\frac{1}{\Delta}\begin{pmatrix} \gamma_2\beta_1 & \gamma_2 \\ \gamma_1 & \gamma_1\beta_2 \end{pmatrix}, \quad \Delta = 1-\beta_1\beta_2$$

Note that if $\Pi = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix}$, then indirect least squares estimates of β_1, β_2 would be $\hat{\pi}_{11}/\hat{\pi}_{12}$ and $\hat{\pi}_{22}/\hat{\pi}_{21}$, respectively. Both parameters are <u>exactly - identified</u>. ▨

Now we are ready to deal with identification

Recall $A = \begin{bmatrix} B \\ \Gamma \end{bmatrix}$. Consider identification for the first equation, which has parameters $A_1 = \begin{bmatrix} B_1 \\ \Gamma_1 \end{bmatrix}$.

Recall $\underset{K\times m}{\Pi} = -\underset{K\times m}{\Gamma}\,\underset{m\times m}{B^{-1}}$ so that $\underset{K\times m}{\Pi}\,\underset{m\times m}{B} + \underset{K\times m}{\Gamma} = \underset{K\times m}{0}$.

This holds for all m equations, so that $\Pi B_1 + \Gamma_1 = 0$ is also true. Write this as

$$[\underset{K\times m}{\Pi} \quad \underset{K\times K}{I_K}]\,\underset{(K+m)\times 1}{A_1} = \underset{K\times 1}{0}$$

In addition, we impose g linear restrictions on A_1. We write these as

$$\underset{g\times(m+k)}{\Phi}\ \underset{(m+k)\times 1}{A_1} = \underset{g\times 1}{\phi}$$, where Φ and ϕ are known.

Then $A_1 = \begin{bmatrix} B_1 \\ \Gamma_1 \end{bmatrix}$ is identified iff

$$\underset{(k+g)\times(m+k)}{\begin{bmatrix} \Pi & I_k \\ \hline \multicolumn{2}{c}{\Phi} \end{bmatrix}} A_1 = \begin{bmatrix} 0 \\ \phi \end{bmatrix}$$

has a unique solution for A_1.

<u>Example 4</u>: Continuing examples 1-3, for equation 1 we have the following restrictions for $A_1 = \begin{bmatrix} B_1 \\ \Gamma_1 \end{bmatrix}$.

Denoting the parameter on y_{1t} by β_0 and the parameter on z_{1t} by γ_0 so that

$A_1 = \begin{bmatrix} \beta_0 \\ -\beta_1 \\ -\gamma_0 \\ -\gamma_1 \end{bmatrix}$, we impose $\beta_0 = 1$ and $\gamma_0 = 0$.

This gives $\Phi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $\phi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. ∎

<u>Thm</u>: A_1 is identified if

$$\text{Rank} \underset{(k+g)\times(m+k)}{\begin{bmatrix} \Pi & I_k \\ \hline \multicolumn{2}{c}{\Phi} \end{bmatrix}} = m+k$$

This theorem will be used to formulate a more precise rank condition. But first note that for

$$\operatorname{rank} \underbrace{\begin{bmatrix} \Pi & I_k \\ \Phi & \end{bmatrix}}_{(k+g)\times(m+k)} = m+k$$

to have any chance of holding, we need at least $m+k$ ~~columns~~ rows in $\begin{bmatrix} \Pi & I_k \\ \Phi & \end{bmatrix}$. This gives us the <u>necessary</u> order condition:

<u>Order Condition (1):</u> $g \geq m$

(# linear restrictions) $\geq$ (# endog. variables).

Note that if we normalize so that the β coefficient on the LHS variables in each structural equation are set to 1 (e.g. the coefficient on y_{1t} in Eq. (1) of Example 1 is set to 1; the coefficient on y_{2t} in Eq. (2) of Example 1 is set to 1), then we no longer need to consider this normalization restriction in Φ. Removing this normalization restriction from Φ gives us the order condition: $\underset{\uparrow}{g^{*}}+\overset{\text{normalization}}{1} \geq m$

(# linear restrictions without normalization restriction)

In this framework, g^* now includes only <u>exclusion restrictions</u>. These are the restrictions that set certain parameters to 0, thereby excluding the associated variables from the equation.

<u>Example 5</u>

$$
\begin{aligned}
y_{1t} &= \beta_{12} y_{2t} + \gamma_{11} Z_{1t} + \gamma_{12} Z_{2t} + u_{1t} \\
y_{2t} &= \beta_{21} y_{1t} + \beta_{23} y_{3t} + \gamma_{22} Z_{2t} + \gamma_{23} Z_{3t} + u_{2t} \\
y_{3t} &= \beta_{31} y_{1t} + \gamma_{31} Z_{1t} + u_{3t}
\end{aligned}
$$

$$
B_1 = \begin{pmatrix} \beta_{11} \\ \beta_{12} \\ \beta_{13} \end{pmatrix} = \begin{pmatrix} 1 \\ \beta_{12} \\ 0 \end{pmatrix} \qquad \Gamma_1 = \begin{pmatrix} \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \end{pmatrix} = \begin{pmatrix} \gamma_{11} \\ \gamma_{12} \\ 0 \end{pmatrix}
$$

Sign errors.

We do not include the normalization restriction in g, the # of exclusion restrictions. For equation 1,

$$
\Phi^* = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \phi^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

$\Phi^* A_1 = \phi^*$, where Φ^* does not include the normaliz. restriction.

$g^* =$ # exclusion restrictions $= 2$

$m = 3$

$\underset{g^*}{2} + \underset{\text{normalization restriction}}{1} \geq 3 \Rightarrow$ Order condition satisfied for equation 1. $\blacksquare$

Now, separating g^* into

g_1^*: # exclusion restrictions for endog. variables

g_2^*: # exclusion restrictions for exog. variables,

we have

$$g_1^* + g_2^* + 1 \geq m$$

$$g_2^* + 1 \geq (m - g_1^*)$$

$$\underbrace{g_2^*}_{\text{\# excluded exogenous variables}} \geq \underbrace{(m - g_1^*)}_{\text{\# of included endogenous variables}} - 1$$

This gives us an alternative way of writing the order condition:

<u>Order Condition (2)</u>: $\begin{pmatrix}\text{\# excluded exog.} \\ \text{variables}\end{pmatrix} \geq \begin{pmatrix}\text{\# of included} \\ \text{endog. variables}\end{pmatrix} - 1$

Sometimes you'll see this written as

(# excl. exog. var.) ≥ (# RHS incl. endog. variables).

Those are the same thing since the right hand side of the two inequalities are the same. For instance, $y_{1t} = \beta_1 y_{2t} + \gamma_1 z_{1t} + u_t$ gives

included endog. variables = 2 (y_{1t} and y_{2t})

RHS incl. endog. variables = 1 (only y_{2t}).

Example 6: For Example 1, Eq. (1), we have

2 included endog. variables, 1 excluded exog. variable

so $1 \geq 2-1=1$, order condition is satisfied.

For Example 5, let r = # included endog. variables
s = # excluded exog. variables

Then Eq. (1): $r=2$, $s=1$ Order Condition Satisfied
Eq. (2): $r=3$, $s=1$ Order Condition fails
Eq. (3): $r=2$, $s=2$ Order Condition Satisfied.

Now lets turn back to the rank condition.

Requiring $\underset{(k+g)\times(m+k)}{\text{rank}\begin{bmatrix} \Pi & I_k \\ \multicolumn{2}{c}{\Phi} \end{bmatrix}} = m+k$ reduces to

requiring $\text{rank}(\Phi A) = m$. Lets see why:

$$\underbrace{\begin{bmatrix} \overset{k\times m}{\Pi} & \overset{k\times k}{I} \\ \multicolumn{2}{c}{\underbrace{\Phi}_{g\times(m+k)}} \end{bmatrix}}_{(k+g)\times(m+k)} \underbrace{\begin{bmatrix} \overset{m\times m}{B} & 0 \\ \underset{k\times m}{\Gamma} & \underset{k\times k}{I} \end{bmatrix}}_{\text{nonsingular}} = \begin{bmatrix} 0 & I \\ \Phi A & \Phi\begin{bmatrix} 0 \\ I \end{bmatrix} \end{bmatrix}$$

Since the second matrix on the LHS is nonsingular, the product of the two matrices has the same rank as the first matrix on the LHS.

Now, $\operatorname{rank}\begin{pmatrix} \Pi & I \\ \Phi & \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 0 & I \\ \Phi A & \Phi\begin{bmatrix} 0 \\ I \end{bmatrix} \end{pmatrix} = \operatorname{rank}(\Phi A) + k$

<u>Rank Condition</u>: $\operatorname{rank}(\Phi A) = m$

Identifiability:

- If the order or rank condition fails, the equation is <u>not identified</u>.
- If the order condition holds with equality, the equation is <u>just identified</u> if the rank condition holds.
- If the order condition holds with inequality, and the rank condition holds, the equation is <u>over-identified</u>.

The order condition is a necessary condition for identifiability.

The rank condition is necessary and sufficient.

<u>Example 7</u>: Discuss identifiability of Example 5.

Here, $A = \begin{pmatrix} 1 & \beta_{21} & \beta_{31} \\ \beta_{12} & 1 & 0 \\ 0 & \beta_{23} & 1 \\ \gamma_{11} & 0 & \gamma_{31} \\ \gamma_{12} & \gamma_{22} & 0 \\ 0 & \gamma_{23} & 0 \end{pmatrix}$

<u>Eq. (1)</u>: From Ex. 6, order condition satisfied with equality.

normalization

$\overline{\Phi} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$; $\overline{\Phi} A = \begin{pmatrix} 1 & \beta_{21} & \beta_{31} \\ 0 & \beta_{23} & 1 \\ 0 & \gamma_{23} & 0 \end{pmatrix}$

restriction   restriction

$\text{rank}(\overline{\Phi} A) = 3 = m$ for $\gamma_{23} \neq 0$.

Eq. (1) is just-identified.

<u>Eq. (2)</u>: From Ex. 6, order condition fails. No need to check

Eq. (2) is not-identified.

<u>Eq. (3)</u>: $\overline{\Phi} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$; $\overline{\Phi} A = \begin{pmatrix} \beta_{12} & 1 & 0 \\ 0 & \beta_{23} & 1 \\ \gamma_{12} & \gamma_{22} & 0 \\ 0 & \gamma_{23} & 0 \end{pmatrix}$

$\text{rank}(\overline{\Phi} A) = 3 = m$

From Ex. 6, order condition holds with inequality.

Eq. (3) is over-identified $\boxtimes$.