2.2. KRONECKER PRODUCTS

Many of the results derived later can be expressed neatly and succinctly in terms of the Kronecker product of matrices. Rather than cover this in the Appendix the definition and some of the properties of this product will be reviewed in this section.

DEFINITION 2.2.1. Let $A=(a_{ij})$ be a $p\times q$ matrix and $B=(b_{ij})$ be an $r\times s$ matrix. The Kronecker product of A and B, denoted by $A\otimes B$, is the $pr\times qs$ matrix

$$A\otimes B=\begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1q}B \\ a_{21}B & a_{22}B & \dots & a_{2q}B \\ \vdots & & & \\ a_{p1}B & a_{p2}B & \dots & a_{pq}B \end{bmatrix}.$$

The Kronecker product is also often called the *direct product*; actually the connection between this product and the German mathematician Kronecker (1823–1891) seems rather obscure.

An important special Kronecker product, and one which occurs often is the following: If B is an $r\times s$ matrix then the $pr\times ps$ block-diagonal matrix with B occurring p times on the diagonal is $I_p\otimes B$; that is

$$I_p\otimes B=\begin{bmatrix} B & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ \vdots & & & \\ 0 & 0 & & B \end{bmatrix}.$$

Some of the important properties of the Kronecker product are now summarized.

(a) $(\alpha A)\otimes(\beta B)=\alpha\beta(A\otimes B)$ for any scalars α,β.

(b) If A and B are both $p\times q$ and C is $r\times s$, then

$$(A+B)\otimes C=A\otimes C+B\otimes C.$$

(c) $(A\otimes B)\otimes C=A\otimes(B\otimes C)$.

(d) $(A\otimes B)'=A'\otimes B'$.

(e) If A and B are both $m\times m$ then

$$\operatorname{tr}(A\otimes B)=(\operatorname{tr}A)(\operatorname{tr}B).$$

(f) If A is $m\times n$, B is $p\times q$, C is $n\times r$, and D is $q\times s$ then

$$(A\otimes B)(C\otimes D)=AC\otimes BD.$$

(g) If A and B are nonsingular then

$$(A\otimes B)^{-1}=A^{-1}\otimes B^{-1}.$$

(h) If H and Q are both orthogonal matrices, so is $H\otimes Q$.

(i) If A is $m\times m$, B is $n\times n$ then

$$\det(A\otimes B)=(\det A)^{n}(\det B)^{m}.$$

(j) If A is $m\times m$ with latent roots $a_1,\dots,a_m$ and B is $n\times n$ with latent roots $b_1,\dots,b_n$ then $A\otimes B$ has latent roots a_ib_j $(i=1,\dots,m;\ j=1,\dots,n)$.

(k) If $A>0$, $B>0$ (i.e., A and B are both positive definite) then $A\otimes B>0$.

These results are readily proved from the definition and are left to the reader to verify. A useful reference is Graybill (1969), Chapter 8.

Now recall the vec notation introduced in (21) of Section 1.2; that is, if $T=[\mathbf{t}_1\mathbf{t}_2\dots\mathbf{t}_q]$ is a $p\times q$ matrix then

$$\operatorname{vec}(T)=\begin{pmatrix}\mathbf{t}_1\\ \mathbf{t}_2\\ \vdots\\ \mathbf{t}_q\end{pmatrix}\qquad(pq\times 1).$$

The connection between direct products and the vec of a matrix specified in the following lemma is often useful. The proof is straightforward (see Problem 2.12).

LEMMA 2.2.2. If B is $r\times m$, X is $m\times n$, and C is $n\times s$ then

$$\operatorname{vec}(BXC)=(C'\otimes B)\operatorname{vec}(X).$$

As an application of this lemma, suppose that X is an $m\times n$ random matrix whose columns are independent $m\times 1$ random vectors, each with the same covariance matrix Σ. That is,

$$X=[\mathbf{X}_1\dots\mathbf{X}_n]$$

where $\text{Cov}(\mathbf{X}_i)=\Sigma$, $i=1,\dots,n$. We then have

$$\text{vec}(X)=\begin{pmatrix}\mathbf{X}_1\\ \vdots\\ \mathbf{X}_n\end{pmatrix}$$

and since the $\mathbf{X}_i$ are all independent with the same covariance matrix it follows that

$$\text{(1)} \qquad \text{Cov}[\text{vec}(X)]=\begin{bmatrix}\Sigma & 0 & \cdots & 0\\ 0 & \Sigma & \cdots & 0\\ \vdots & & & \\ 0 & 0 & & \Sigma\end{bmatrix} \qquad (mn\times nm)$$
$$=I_n\otimes\Sigma.$$

Now suppose we transform to a new random matrix Y given by $Y=BXC$, where B and C are $r\times m$ and $n\times s$ matrices of constants. Then $E(Y)=BE(X)C$ and, from Lemma 2.2.2,

$$\text{vec}(Y)=(C'\otimes B)\,\text{vec}(X)$$

so that

$$E[\text{vec}(Y)]=(C'\otimes B)E[\text{vec}(X)].$$

Also, using (3) of Section 1.2,

$$\begin{aligned}\text{Cov}(\text{vec}(Y))&=(C'\otimes B)\,\text{Cov}[\text{vec}(X)](C'\otimes B)'\\ &=(C'\otimes B)(I_n\otimes\Sigma)(C\otimes B')\\ &=C'C\otimes B\Sigma B',\end{aligned}$$

where we have used (1) and properties (d) and (f), above.

Some other connections between direct products and vec are summarized in the following lemma due to Neudecker (1969), where it is assumed that the sizes of the matrices are such that the statements all make sense.

LEMMA 2.2.3.

(i) $\mathrm{vec}(BC)=(I\otimes B)\mathrm{vec}(C)=(C'\otimes I)\mathrm{vec}(B)=(C'\otimes B)\mathrm{vec}(I)$

(ii) $\mathrm{tr}(BCD)=(\mathrm{vec}(B'))'(I\otimes C)\mathrm{vec}(D)$

(iii) $\mathrm{tr}(BX'CXD)=(\mathrm{vec}(X))'(B'D'\otimes C)\mathrm{vec}(X)$
$\qquad =(\mathrm{vec}(X))'(DB\otimes C')\mathrm{vec}(X)$

Proof. Statement (i) is a direct consequence of Lemma 2.2.2. Statement (ii) is left as an exercise (Problem 2.13). To prove the first line of statement (iii), write

$$\begin{aligned}\mathrm{tr}(BX'CXD)&=\mathrm{tr}(BX')C(XD)\\&=(\mathrm{vec}(XB'))'(I\otimes C)\mathrm{vec}(XD)\quad\text{using (ii)}\\&=[(B\otimes I)\mathrm{vec}(X)]'(I\otimes C)(D'\otimes I)\mathrm{vec}(X)\quad\text{using (i)}\\&=(\mathrm{vec}(X))'(B'\otimes I)(I\otimes C)(D'\otimes I)\mathrm{vec}(X)\quad\text{using property (d)}\\&=\mathrm{vec}(X)'(B'D'\otimes C)\mathrm{vec}(X)\quad\text{using property (f)}.\end{aligned}$$

The second line of statement (iii) is simply the transpose of the first.

PROBLEMS

2.1. If X is $n\times m$ and Y is $m\times p$ prove that

$$d(XY)=X.dY+dX.Y.$$

2.2. Prove Theorem 2.1.7.

2.3. Prove that if X, Y and B are $m\times m$ lower-triangular matrices with $X=YB$ then

$$(dX)=\prod_{i=1}^{m} b_{ii}^{m+1-i}(dY).$$

2.4. Show that if $X=Y+Y'$, where Y is $m\times m$ lower-triangular, then

$$(dX)=2^m(dY).$$