$$y_1 = \beta y_2 + \varepsilon_1 = \beta Z \Pi_2 + v_1$$

$$y_2 = Z\Pi_2 + v_2$$

$$\hat{\beta}_{2SLS} - \beta = \frac{\sum_{i=1}^{n} (v_{1i} - \beta z_i' (\hat{\Pi}_2 - \Pi_2)) \cdot z_i' \hat{\Pi}_2}{\sum_{i=1}^{n} (z_i' \hat{\Pi}_2)^2}$$

$$= \frac{\sum_{i=1}^{n} (v_{1i} - \beta z_i' (\hat{\Pi}_2 - \Pi_2)) \cdot z_i' \hat{\Pi}_2}{\underbrace{R_f^2 \sum_{i=1}^{n} y_{2i}^2}_{\text{Hahn proved this}}}$$

theoretical 1st stage R^2 → R_f^2; $\sum y_{2i}^2$ ← variance of y_2.

$$E\left[\sum_{i=1}^{n} (v_{1i} - \beta z_i'(\hat{\Pi}_2 - \Pi_2)) z_i' \hat{\Pi}_2\right] = K\sigma_{\varepsilon v_2}$$ ◦ Cochrane's theorem.

$$E\left[\sum_{i=1}^{n} (z_i' \hat{\Pi}^2\right] = n\Pi_2' R \Pi_2 + K\sigma_{v_2 v_2}$$

$$E[v_2' P_z v_2] = K\sigma_{v_1 v_2}$$

$$R = E[Z'Z/n]$$

$$\Rightarrow E[\hat{\beta}_{2SLS}] - \beta = \frac{K\sigma_{\varepsilon v_2}}{R_f^2} \frac{1}{\sum_{i=1}^{n} y_{2i}^2} \approx \frac{K\sigma_{\varepsilon v_2}}{n\Pi_2' R\Pi_2 + K\sigma_{v_2 v_2}}$$

$$E[\hat{\beta}_{OLS} - \beta] = \frac{Cov(y_2, \varepsilon)}{Var(y_2)} \approx \frac{\sigma_{\varepsilon v_2}}{\Pi_2' R \Pi_2 + \sigma_{v_2 v_2}}$$

No identification

$$\Pi_2 = 0$$

$$E[\hat{\beta}_{2SLS} - \beta] \approx \frac{\sigma_{\varepsilon v_2}}{\sigma_{v_2 v_2}}$$

$$\text{and } E[\hat{\beta}_{OLS} - \beta] \approx \frac{\sigma_{\varepsilon v_2}}{\sigma_{v_2 v_2}}$$

If they were different, then you could back out β.
- Then β is identified $\cancel{\rightarrow}$.

Local Non-ID

$$\pi_2 = \frac{a}{\sqrt{n}} \rightarrow 0$$

$$E[\hat{\beta}_{2SLS}] - \beta \approx \frac{K\sigma_{\varepsilon v_2}}{a'Ra + K\sigma_{v_2 v_2}} = \frac{\sigma_{\varepsilon v_2}}{\frac{1}{K}a'Ra + \sigma_{v_2 v_2}}$$

$$E[\hat{\beta}_{OLS}] - \beta \approx \frac{\sigma_{\varepsilon v_2}}{\frac{1}{n}a'Ra + \sigma_{v_2 v_2}}$$

2SLS is smaller than OLS if $K < n$ (less biased).
- Most of the time, 2SLS has lower MSE.

Bias corrected 2SLS (Nagar)

$$\sigma_{\varepsilon v_2} = E\left[\frac{1}{N-K}\underbrace{(y_2' Q_z)}_{\hat{v}_2'}\underbrace{(y_1 - y_2\beta)}_{\hat{\varepsilon}}\right] \quad \text{where } Q_z \equiv I - P_z$$

$$\Rightarrow E[\hat{\beta} - \beta] \approx \frac{K\sigma_{\varepsilon v_2}}{y_2' P_z y_2} \approx E\left[\frac{M}{N-K}\hat{v}_2'\hat{\varepsilon}\right] = E[dq'(y_1 - y_2\beta)]$$

$$M = \frac{K}{y_2' P_z y_2}$$

Bad news about Nagar: doesn't have any moments

$$\hat{\beta}_N = \frac{\frac{y_2' P_Z y_1}{y_2' P_Z y_2} - \frac{K}{N-K}\frac{y_1' Q_Z y_2}{y_2' P_Z y_2}}{1 - \frac{K}{N-K}\frac{y_2' Q_Z y_2}{y_2' P_Z y_2}} = \frac{y_2' P_Z y_1 - \frac{K}{N-K} y_1' Q_Z y_2}{y_2' P_Z y_2 - \frac{K}{N-K} y_2' Q_Z y_2}$$

Denominator is zero when $\Pi_2 = 0$.

If Π_2 is close to zero, still will not have moments by continuity.

$$\underbrace{E\left[y_2' P_Z y_2 - \frac{K}{N-K}(y_2' Q_Z y_2)\right)}_{} = K\sigma_{vv} - \frac{K}{N-K}\{(N-K)\sigma_{vv}\} = 0$$

$$= E[y_2' P_Z y_2] - \frac{K}{N-K} E[y_2' Q_Z y_2]$$

$$= tr\, P_Z E[y_2 y_2'] - \frac{K}{N-K} tr\, Q_Z E[y_2' y_2]$$

$$= K\sigma_{vv} - \frac{K}{N-K}(N-K)\sigma_{vv} = 0$$

LIML

$$\hat{\beta}_{LIML} = \frac{y_2' P_Z y_1 - \theta y_1' Q_Z y_2}{y_2' P_Z y_2 - \theta y_2' Q_Z y_2}$$

K class form

$$\theta = \frac{\lambda}{1-\lambda}$$

λ is smallest eit of $|y_2' P_Z y_2 - \lambda y_2' y_2|$

$E[\lambda] = \frac{K}{N}$

$E[y_2' P_Z y_2 - \theta (y_2' Q_Z y_2)] = 0$

Fuller takes $\varphi = \theta - \frac{\alpha}{N-K}$

$\alpha = 1 \Rightarrow$ unbiased
$\alpha = 4 \Rightarrow$ min RMSE

$$\hat{\beta}_{FULLER} = \frac{y_2' P_Z y_1 - \varphi y_1' Q_Z y_2}{y_2' P_Z y_2 - \varphi y_2' Q_Z y_2}$$

$E[y_2' P_Z y_2 - \varphi y_2' Q_Z y_2] = K\sigma_{vv} - \varphi(N-K)\sigma_{vv} = \alpha \sigma_{vv}$

2SLS $E[y_2' P_Z y_2] = K\sigma_{vv}$

- Not convincing that Fuller has all moments

Concentration parameter

$$\underbrace{\mu^2}_{\text{concentration parameter}} = \frac{N E[\Pi_2' Z' Z \Pi_2]}{E[V'V]} = \frac{N \Pi_2' E[Z'Z] \Pi_2}{\sigma_{vv}} \approx \frac{NR^2}{1-R^2}$$

$$F = \frac{NR^2}{K(1-R^2)}$$

Don't want to use Nagar or JIVE

Shouldn't use LIML when you have weak instruments.

How might you tell if you have weak instruments?

- Run forward and reverse 2SLS. If they are close, you are in good shape.

$$\hat{\beta}_{2SLS} \equiv \frac{\sum_{i=1}^{n} \hat{y}_{2i} y_{1i}}{\sum_{i=1}^{n} \hat{y}_{2i}^2}$$ Forward regression

$$\hat{c}_{2SLS} \equiv \frac{\sum_{i=1}^{n} y_{1i} \hat{y}_{2i}}{\sum_{i=1}^{n} \hat{y}_{1i}^2}$$ Reverse regression

$$\sqrt{n}\left(\hat{\beta}_{2SLS} - \frac{1}{\hat{c}_{2SLS}}\right) = o_p(1)$$

Specification test

$$B = -\alpha \frac{\theta \sigma_\varepsilon^2 + \alpha \det \Omega}{(\theta + \alpha \omega_{22})(\beta\theta + \alpha \omega_{12})}$$

$$\sqrt{n}\left(\hat{\beta}_{2SLS} - \frac{1}{\hat{c}_{2SLS}} - \hat{B}\right) \xrightarrow{d} N(0, V)$$

$$H_0: \operatorname{plim}\left(\hat{\beta}_{2SLS} - \frac{1}{\hat{c}_{2SLS}} - B\right) = 0$$

Can base this test on Nagar estimator as well.

LIML is the optimal combination of forward and reverse Nagar estimates.

(*) EIV in the instruments doesn't hurt you.

(*) 2SLS is biased towards OLS

$\Rightarrow$ reverse 2SLS is biased away from OLS.