FOCs from last time:

1 $[T(B')^{-1} - Y'(YB + Z\Gamma)\Sigma^{-1}]^u = 0$

2 $[\quad - Z'(YB + Z\Gamma)\Sigma^{-1}]^u = 0$

3 $[T\Sigma - (YB + Z\Gamma)'(YB + Z\Gamma)]^u = 0$

We will do covariance restrictions. Without restrictions,

$$\hat{\Sigma} = \frac{u'u}{T}$$

Let some $\sigma_{ij} = 0$

$$\Rightarrow [T(B')^{-1}\Sigma - Y'(YB + Z\Gamma)\Sigma^{-1}]^u = 0$$

Let N_j be a set of indices m for the j^{th} row of Σ

s.t. $\sigma_{jm} = 0$

$$[T(B')^{-1}\Sigma]_{ij} = T \sum_{k=1}^{M} \underbrace{\beta^{ki}}_{k,i^{th} \text{ elt of } (B')^{-1}} \sigma_{kj} = (V_i' - \sum_{k \in N_j} \beta^{ki} u_k')u_j \text{ only}$$

for zero elements

$$\Rightarrow [T(B')^{-1}\Sigma - Y'(YB + Z\Gamma)\Sigma^{-1}]_{ij} = [V_i - \sum_{k \in N_j} \beta^{ki} u_k - Z\Pi_i - V_i]' u_j = 0$$

$$= -[Z\Pi_i + b_{ji}'\hat{u}_i]'u_j$$

The FOC becomes

$$-\{[(B^{-1})'\Gamma'Z' + \underbrace{\hat{V}'}_{\text{uncorrelated residuals multiplied by } B^{-1}})(YB + Z\Gamma)]\Sigma^{-1}\}^u = 0$$

replaced Y_i by r.f.

- Imposing covariance restrictions in 3SLS does not improve asymptotic efficient.

Hausman, Newey, Taylor:

$\underbrace{A3SLS}_{\text{augmented 3SLS}} \overset{A}{=} FIML > 3SLS$

- can use $\hat{u}_j$ as instrument if $\sigma_{ij}=0$.

LIML

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & I\otimes Z \end{bmatrix}\begin{bmatrix} \delta_1 \\ \Pi_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ v_2 \end{bmatrix} \qquad V\begin{bmatrix} u_1 \\ v_2 \end{bmatrix} = \Psi\otimes I = \begin{bmatrix} \sigma_{11} & \Psi_{12} \\ & \Omega_{22} \end{bmatrix}\otimes I$$

- assume that eqn 1 is overidentified and that other eqns are just identified. Then FIML gives us LIML for eqn 1

FOCs:

$$\left[\begin{pmatrix} (B')^{-1}\Gamma' Z' \\ -Z' \end{pmatrix}\begin{pmatrix} YB_1 + Z\Gamma_1 \\ Y_2 - Z\Pi_2 \end{pmatrix}\Psi^{-1}\right]^u = 0 \qquad \text{FIML FOCs}$$

- $B_{11}=1 \Rightarrow \log[\det(B)^{-1}] = \log 1 = 0$
- It is as if B is triangular

Stack the FOCs: Recall that 3SLS ≡ 2SLS in this setup (since eqn 1 is over id and others are just id)

$$\Rightarrow \hat{\delta}_1 = (\hat{X}_1'X_1)^{-1}\hat{X}_1'y_1 \qquad \hat{X}_1 = [Z\underbrace{\hat{\Pi}_2}_{2SLS}\ Z_1]$$

$$Z\hat{\Pi}_2 = P_Z y_1$$

$$\hat{\Pi}_2 = (I\otimes(Z'Z)^{-1}Z')y_2 - [\hat{\Psi}_{21}\hat{\sigma}^{11}\otimes(Z'Z)^{-1}Z'](y_1 - X_1\hat{\delta}_1)$$

$\Rightarrow$ LIML is just a special case of FIML

Since plim $\hat{\Pi}_2^{LIML}$ = plim $\hat{\Pi}_{2SLS}$

- For any bowl-shaped loss function, up to second-order, LIML is better than 2SLS.
- LIML is median-unbiased $\Rightarrow$ approximately mean-unbiased

$$\text{plim}\sqrt{T}(\hat{\delta}_{LIML}-\hat{\delta}_{2SLS}) = \text{plim}\left(\frac{\hat{X}_{1,LIML}'X_1}{T}\right)^{-1}\frac{\hat{X}_{1,LIML}'u_1}{\sqrt{T}} - \text{plim}\left(\frac{\hat{X}_{1,2SLS}'X_1}{T}\right)^{-1}\frac{\hat{X}_{1,2SLS}'u_1}{\sqrt{T}}$$

$$= \text{plim}\left(\frac{1}{T}\hat{X}_{1,2SLS}'X_1\right)^{-1}\underbrace{\sqrt{T}(\hat{\Pi}_{2,LIML}-\hat{\Pi}_{2,unrest.}\ I_1)}_{O_p(1)}\underbrace{\left(\frac{Z'u_1}{T}\right)}_{o_p(1)}$$

$$= 0$$

LIML as a k-class estimator

$$\hat{X}_1 = [\hat{Y}_1 \quad Z_1] = [((1-k)I + kP_Z)Y_1 \quad Z_1]$$

LIML: $k = \frac{1}{1-\hat{\lambda}}$ where $\hat{\lambda}$ is smallest char. value of $0 = |X_1'P_Z X_1 - \lambda X_1'X_1|$

- For consistency, need plim $k = 1$
- For efficiency, need plim $\sqrt{T}(k-1) = 0$

How to test overidentifying restrictions?

Full information $\underset{\text{\# of overid.}}{W_p} = -2 \log\left[\frac{L(\hat{\Pi}_R, \hat{\Omega}_R)}{L(\hat{\Pi}_{UR}, \hat{\Omega}_{UR})}\right]$

$$= T \log\left[\frac{\det \hat{\Omega}_R}{\det \hat{\Omega}_{UR}}\right] \xrightarrow{d} \chi^2(p)$$

where $\hat{\Pi}_R = \hat{\Gamma}_F \hat{B}_F^{-1}$

3SLS: $W_p = \hat{u}_{3SLS}' (\hat{\Sigma}_{3SLS}^{-1} \otimes P_Z) \hat{u}_{3SLS} \xrightarrow{d} \chi^2(p)$

Limited information Let p_1 = degree of overid for eqn 1.

LIML: $T(\hat{k}-1) \sim \chi^2(p_1)$

2SLS: $\hat{\lambda} \approx \frac{(y_1 - X_1\hat{\delta}_1)' P_Z (y_1 - X_1\hat{\delta}_1)}{\hat{\sigma}_{11}} \sim \chi^2(p_1)$

$= TR^2$

Omnibus test $\frac{Z'\varepsilon}{\sqrt{T}}$ $V = \left(\frac{1}{T}(Z'\Omega Z)\right)$

Test $W = \frac{1}{T} \hat{\varepsilon}' Z V^{-1} Z' \hat{\varepsilon} \overset{A}{\sim} \chi^2(p_1)$

- This will work for conditional heteroskedasticity

Hausman test

$\tilde{Z}_1$ – confident that those are instruments

$\tilde{\tilde{Z}}_1$ – not sure

$\tilde{\delta}_1$ uses just $\hat{Z}_1$

$\hat{\delta}_1$ uses both $\hat{Z}_1$ and $\hat{\tilde{Z}}_1$

$\hat{q} = \tilde{\delta}_1 - \hat{\delta}_1$

Let $W = \frac{1}{\hat{\sigma}_{11}} \hat{q}' \left[(X_1' P_{\tilde{Z}_1} X_1)^{-1} - (X_1 P_Z X_1)^{-1} \right] \hat{q}$

dof $= \min\{p_1, k_1\}$

- Biased towards rejection.

Nonlinear

$f_1(y, Z, \alpha) \overset{\text{separable}}{=} u_1$ nonseparable: $f_1(y, Z, \alpha, u_1) = 0$

- Matzkin

$f_1(y, Z, \alpha) \approx \underbrace{f_1(y, Z, \alpha')}_{= u_1'} + \frac{\partial f_1}{\partial \alpha} \cdot (\alpha - \alpha')$

$= g_1' \cdot (\alpha - \alpha') + u_1'$

It might be the case that plim $\frac{g_1' u_1}{N} \neq 0$

Let $\hat{\theta} = \underbrace{\alpha^2 - \alpha'}_{\text{difference after iterating}} = (\underbrace{W_1}_{\text{true IV}}' g_1)^{-1} W_1' \hat{u}_1$

Can prove that $\sqrt{N}(\hat{\alpha} - \alpha) \xrightarrow{d} X$,

where $X \sim N(0, \text{plim}\, \hat{\sigma}_{11} (W_1' g_1(\hat{\alpha}))^{-1} (W_1' W_1) (g_1'(\hat{\alpha}) W_1)^{-1})$

Forbidden regression:

$\min_{\alpha} \Sigma u_{1i}^2 = \min_{\alpha} \Sigma f_1(y_{1i}, \hat{y}_{1i}, Z_{1i}, \alpha)$

$y_1 = \beta_1 y_2^{\beta_2} + Z\gamma_1 + u_1$

spse we replace y_2 with $\hat{y_2}$

$\Rightarrow y_1 = \beta_1 (\hat{y_2} + \hat{v_2})^{\beta_2} + Z\gamma_1 + u_1$

$\approx \beta_1 \hat{y_2}^{\beta_2} + Z_1\gamma_1 + u_1 + \beta_1 \log(\hat{y_2}) \hat{y_2}^{\beta} \hat{v_2} + \beta_1 \log^2(\hat{y_2}) \hat{y_2}^{\beta_2} \hat{v_2}^2$

In the linear case:

$E[\hat{y}_1] = E[Z\hat{\Pi} + \hat{v}] = E[Z\hat{\Pi}]$

Nonlinear simultaneous equation systems are almost always identified, because if you have one exogenous variable, you can take all nonlinear functions of it to get infinite instruments

$\alpha^2 = \alpha^1 + (\hat{W}_1 g_1)^{-1} \hat{W}_1' f_1(y, Z, \alpha^1)$

$= \alpha^1 + (g_1' \hat{W}_1 (\hat{W}_1' \hat{W}_1)^{-1} \hat{W}_1' g_1')^{-1} g_1 \hat{W}_1 (\hat{W}_1' \hat{W}_1)^{-1} \hat{W}_1' f(y, Z, \alpha^1)$

$\hat{W}_1 = P_{W_1} g_1'$

amemiya's nonlinear 2SLS (NL2SLS)

$J = \min_{\alpha} f_1(y, Z, \alpha) \hat{W} (\hat{W}' \hat{W})^{-1} \hat{W}' f(y, Z, \alpha) \Rightarrow \hat{J} \overset{A}{\sim} \chi^2(p)$

$\sqrt{N}(\hat{\alpha} - \alpha) \xrightarrow{d} N(0, \text{plim } \frac{1}{N} \sigma_{11} (g_1' P_{\hat{W}_1} g_1)^{-1})$

Let $f(y, Z, \alpha) = u$ $\quad V(u) = \Sigma \otimes I$

can then do NL3SLS