$y = \tilde{X}\beta + \varepsilon$

FGLS $\hat{\Sigma}$ $\hat{\hat{A}}$ GLS Σ

$\text{plim}\,\sqrt{T}\,(\hat{\hat{\beta}}(\hat{\varepsilon}) - \hat{\beta}(\varepsilon)) = 0$

$\text{Var}(\hat{\beta}_{OLS}) = (\tilde{X}'\tilde{X})^{-1}\tilde{X}'\Omega\tilde{X}(\tilde{X}'\tilde{X})^{-1} \geq \text{Var}(\hat{\beta}_{GLS})$

$\hat{\beta}_{OLS} \equiv \hat{\hat{\beta}}(\hat{\Sigma}) = \hat{\beta}(\Sigma)$ if $\tilde{X} = I \otimes X$

$$\underset{(K_1+K_2)\times 1}{\text{plim}\begin{bmatrix}\hat{\hat{\beta}}_1 - \beta_1 \\ \hat{\hat{\beta}}_2 - \beta_2\end{bmatrix}} = \left\{\frac{1}{T}\begin{bmatrix}X_1' & 0 \\ 0 & X_2'\end{bmatrix}\begin{bmatrix}\hat{\sigma}^{11}I_T & \hat{\sigma}^{12}I_T \\ \hat{\sigma}^{21}I_T & \hat{\sigma}^{22}I_T\end{bmatrix}\begin{bmatrix}X_1 & 0 \\ 0 & X_2\end{bmatrix}\right\}^{-1} \cdot \left\{\frac{1}{T}\begin{bmatrix}X_1' & 0 \\ 0 & X_2'\end{bmatrix}\underbrace{\begin{bmatrix}\hat{\sigma}^{11}I_T & \hat{\sigma}^{12}I_T \\ \hat{\sigma}^{21}I_T & \hat{\sigma}^{22}I_T\end{bmatrix}}_{(\hat{\Sigma}\otimes I_T)^{-1}}\begin{bmatrix}\varepsilon_1 \\ \varepsilon_2\end{bmatrix}\right\} \qquad (1)$$

If some equation is misspecified, $\text{plim}\,\frac{1}{T}X_i'\varepsilon_i \neq 0$

$$(1) = M^{-1}\left\{\text{plim}\,\frac{1}{T}\begin{bmatrix}\hat{\sigma}^{11}\overbrace{X_1'\varepsilon_1}^{\neq 0} + \hat{\sigma}^{12}X_1'\varepsilon_2 \\ \hat{\sigma}^{12}\underbrace{X_2'\varepsilon_1}_{\neq 0} + \hat{\sigma}^{22}\underbrace{X_2'\varepsilon_2}_{=0}\end{bmatrix}\right\}$$

if $\text{plim}\,\frac{X_1'\varepsilon_1}{T} \neq 0$ $\Rightarrow$ get inconsistent estimators for both equations

How do you test if this is going to be consistent?

$y_1 = X_1\beta_{11} + \tilde{X}_2\beta_{12} + u_2$ ← assumed $\beta_{12} = 0$

$y_2 = \tilde{X}_1\beta_2 + X_2\beta_{22} + u_2$ ← $\beta_2 = 0$

} This is seemingly unrelated regression

$H_0: \beta_{12} = 0$
$\quad\;\; \beta_{21} = 0$

Wald test:

Estimate $\hat{\Pi}$. Unrestricted form

$$\underbrace{\hat{\Pi}}_{\substack{\text{OLS} \\ \text{eqn by eqn}}} = [I \otimes (Z'Z)^{-1} Z'] y$$

$$\sqrt{T}(\hat{\Pi} - \Pi) \xrightarrow{d} N(0, \Omega \otimes (Z'Z)^{-1})$$

$$y_1 = \frac{\gamma_1}{\Delta} z_1 + \frac{\beta_1 \gamma_2}{\Delta} z_2 + \frac{\beta_1 \gamma_3}{\Delta} z_3 + v_1$$

$$y_2 = \frac{\beta_1 \gamma_1}{\Delta} z_1 + \frac{\gamma_2}{\Delta} z_2 + \frac{\gamma_3}{\Delta} z_3 + v_2$$

overidentified, but restrictions are all nonlinear.

$H_0: h(\Pi) = 0$ (all three of $\frac{\beta_1\gamma_1}{\Delta} / \frac{\gamma_1}{\Delta}$, $\frac{\beta_1\gamma_2}{\Delta} / \frac{\gamma_2}{\Delta}$, $\frac{\beta_1\gamma_3}{\Delta} / \frac{\gamma_3}{\Delta}$ are equal)

$$h(\hat{\Pi}) \approx \overbrace{h(\Pi)}^{=0} + \overbrace{\frac{\partial h}{\partial \Pi}}^{\text{matrix}} (\hat{\Pi} - \Pi) + R$$

$$\Rightarrow h(\hat{\Pi}) \approx \frac{\partial h}{\partial \Pi} (\hat{\Pi} - \Pi) = H'(\hat{\Pi} - \Pi)$$

should be transpose

$$\Rightarrow \sqrt{T}(h(\hat{\Pi}) - \underbrace{h(\Pi)}_{=0}) = H' \sqrt{T}(\hat{\Pi} - \Pi) \xrightarrow{d} N(0, H \Psi H')$$

$$\Rightarrow W = h(\hat{\Pi}) \{ H [\hat{\Omega} \otimes (Z'Z)^{-1}] H' \}^{-1} h(\hat{\Pi}) \overset{A}{\sim} \chi^2 (\text{\# of over identication})$$

$$LR = -2\left[\underbrace{\log L}_{\mathcal{L}}(\hat{\Pi}_{OLS}, \hat{\Sigma}) - \log L(\hat{\hat{\Pi}}_{ML}, \hat{\hat{\Sigma}}_{ML})\right] \overset{A}{\sim} \chi^2(\#OID)$$

where $\mathcal{L} = \text{constant} + \frac{1}{2}\log\det(\Omega^{-1}) - \frac{1}{2}\operatorname{tr}(\Omega^{-1}V'V)$

$= \text{constant} + \frac{1}{2}\log\det(\Omega^{-1}\otimes I) - \frac{1}{2}[(y-(I\otimes Z)\pi)' \cdot (\Omega^{-1}\otimes I)(y-(I\otimes Z)\pi)]$

$= \text{constant} - \frac{nT}{2}\log\det\Omega$ concentrating the likelihood form

3SLS = 2SLS + SUR

LI ≡ FI if all eqns are just identified

$y_i = \bar{X}_i\delta_i + u_i$

$$\begin{bmatrix} y_{11} \\ \vdots \\ y_{1T} \\ \vdots \\ y_{m1} \\ \vdots \\ y_{mT} \end{bmatrix} = \begin{bmatrix} \bar{X}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{X}_M \end{bmatrix} \begin{bmatrix} \delta_{11} \\ \vdots \\ \delta_{1,r_1+s_1} \\ \vdots \\ \delta_{m1} \\ \vdots \\ \delta_{m,r_m+s_m} \end{bmatrix} + \begin{bmatrix} u_{11} \\ \vdots \\ u_{1T} \\ \vdots \\ u_{M1} \\ \vdots \\ u_{MT} \end{bmatrix}$$

Can't use OLS due to endogeneity

Looking for:

$\hat{\delta}_{IV} = (\hat{W}'\bar{X})^{-1}\hat{W}'y$ where

$$\hat{W} = \begin{bmatrix} W_1 & & 0 \\ & \ddots & \\ 0 & & W_M \end{bmatrix}$$

If $W_i = Z(Z'Z)^{-1}Z'\bar{X}_i = P_Z\bar{X}_i$, then this is just 2SLS eqn by equation.

$\Rightarrow \tilde{W}' = \hat{\tilde{X}}(I \otimes P_z)$

$\Rightarrow \hat{\delta}_{2SLS} = (\hat{\tilde{X}}'(I \otimes P_z)^{-1}\tilde{X})^{-1}\hat{\tilde{X}}'(I \otimes P_z)y$

But we also know $V(u) = \Sigma \otimes I_T$

Let $\hat{\delta}_{3SLS} = (\hat{\tilde{X}}'(\underbrace{\hat{\Sigma}^{-1}}_{\text{2SLS first to estimate this}} \otimes P_z)\hat{\tilde{X}})^{-1} \tilde{\delta}(\hat{\Sigma}^{-1} \otimes P_z)y$

Can be shown that $\sqrt{T}(\hat{\delta}_{3SLS} - \hat{\delta}_{FIML}) \xrightarrow{d} 0$

Still need to make sure we have correct specification

FGLS: $\hat{\beta}_{OLS} \to \hat{\Sigma}^1 \to \hat{\beta}_{FGLS}(\hat{\Sigma}^1) \to \hat{\Sigma}^2 \to \hat{\beta}_{FGLS}(\hat{\Sigma}^2) \to \hat{\Sigma}^3 \to \dots$

$\to \hat{\beta}_{FIML}$ (Hellwig)

If you iterate 3SLS, does this converge to $\hat{\delta}_{FIML}$?

- No. $W_i = P_z X_i = (Z\hat{\Pi}_i \; Z_i)$

<u>Why</u> is 3SLS better than 2SLS?

$Var(\hat{\delta}_{2SLS}) = \text{plim}\,(\hat{X}'X)^{-1}\hat{X}'(\Sigma \otimes I)\hat{X}(\hat{X}'X)^{-1}$

$$\begin{bmatrix} (\hat{X}_1'X_1)^{-1} & 0 \\ 0 & (\hat{X}_1'X_1) \end{bmatrix}\begin{bmatrix} \sigma_{11}(\hat{X}_1'\hat{X}_1) & \sigma_{12}(\hat{X}_1'\hat{X}_2) \\ & \sigma_{22}(\hat{X}_2'\hat{X}_2) \end{bmatrix}\begin{bmatrix} (\hat{X}_1'X_1)^{-1} & 0 \\ 0 & (\hat{X}_2'X_2)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11}(\hat{X}_1'\hat{X}_1)^{-1} & \sigma_{12}(\hat{X}_1'X_1)^{-1}(\hat{X}_1'\hat{X}_2)(\hat{X}_2'X_2)^{-1} \\ & \sigma_{22}(\hat{X}_2'\hat{X}_2)^{-1} \end{bmatrix}$$

$Var(\hat{\delta}_{3SLS}) = [\hat{X}'(\Sigma \otimes I)^{-1}\hat{X}]^{-1}$

$Var(\hat{\delta}_{3SLS}) - Var(\hat{\delta}_{2SLS})$

$g'[\{\hat{X}'(\Sigma \otimes I)^{-1}\hat{X}\} - \{[\hat{X}'\hat{X}][\hat{X}'(\Sigma \otimes I)\hat{X}]^{-1}[\hat{X}'\hat{X}]^{-1}\}]g$

Let $\Sigma^{-1} \otimes I = P'P \Rightarrow \Sigma \otimes I = P^{-1}(P^{-1})'$

$\Rightarrow g'\{\hat{X}'P'[I - \underbrace{(P')^{-1}\hat{X}}_{B}[\hat{X}'P^{-1}(P^{-1})'\hat{X}]^{-1}\hat{X}'P^{-1}]P\hat{X}\}g$

$= h'[I - B[B'B]^{-1}B]h \geq 0$

$\hat{\delta}_{3SLS} = (\hat{W}'\hat{W})^{-1}\hat{W}'y \qquad \hat{W} = I \otimes Z$

$\Rightarrow \hat{\delta}_{3SLS} = \hat{\delta}_{2SLS}$

What if $T \leq K$? Can't do 2SLS or 3SLS, since cannot estimate reduced form.

$\hat{\delta}_{OIV}$ is more efficient than $\hat{\delta}_{3SLS}$

$\hat{\delta}_{2SLS(1)} = (W_1'X_1)^{-1}W_1'y_1$

can choose subset of instruments and still get consistent estimate

$\Rightarrow$ IV for each eqn $\begin{bmatrix} \hat{\delta}_1 \\ \vdots \\ \hat{\delta}_M \end{bmatrix} \to \hat{\Sigma}$

$\searrow \hat{B}, \hat{\Gamma} \to \hat{\hat{\Pi}} = -\hat{\Gamma}\hat{B}^{-1} \to \hat{W}_i = [Z\hat{\hat{\Pi}}_i \; Z_i]$

where $\hat{\hat{\Pi}}_i \to \Pi_i$ and $\hat{\Pi}_i \to \Pi_i$ unless we have any misspecification. asymptotically equivalent to $\hat{\delta}_{3SLS}$