Problem set to be posted soon. Due in three weeks.
- Should all read Greene and Wooldridge.

Three difficult topics:
- Hausman-Taylor stuff
  - favorite question for the final (or general)
  - will be covered next week. Read this paper (EMA 83)
- FIML
- Weak instruments.

Identification

Strategies: $\boxed{1}$ Normalizations: $\underbrace{\alpha}_{=1 \text{ by normalization}} y = x\beta + \varepsilon$

$\boxed{2}$ Exclusion restrictions: $y_1 = x\beta + z_1\lambda + z_2 \cdot 0 + \varepsilon$

$y_2 = x\hat{\beta} + z_2\delta + z_1 \cdot 0 + v$

$\boxed{3}$ Linear restrictions: $\beta_1 + \beta_2 = 1$
- comes from theory

$\boxed{4}$ Covariance restrictions: $\sigma_{12} = 0$

Raersol $\boxed{5}$ "Lagos": (Reason)
- higher order moment restrictions
- nonlinear restrictions
- intertemporal restrictions

Model: $\underset{T\times M}{Y}\ \underset{M\times M}{B} + \underset{T\times K}{Z}\ \underset{K\times M}{\Gamma} = \underset{T\times M}{U}$

M equations - endo. variable
K exogenous variables
T observations

$$[Y_{1t}\ Y_{2t}]\begin{bmatrix}\beta_{11} & \beta_{21}\\ \beta_{12} & \beta_{22}\end{bmatrix}+[Z_{1t}\ Z_{2t}\ Z_{3t}]\begin{bmatrix}\gamma_{11} & \gamma_{21}\\ \gamma_{12} & \gamma_{22}\\ \gamma_{13} & \gamma_{23}\end{bmatrix}=[u_{1t}\ u_{2t}]$$

for $t=1,\dots,T$

Reduced Form: $Y=-Z\Gamma B^{-1}+UB^{-1}$

$$=\underset{T\times K}{Z}\ \underset{K\times M}{\Pi}+\underset{T\times M}{V} \qquad \Pi=-\Gamma B^{-1},\ V=UB^{-1}$$

$E[Y]=E[Z]=0$

$\mathrm{Var}(u_t)=\underset{M\times M}{\Sigma}$ disturbances across equations

$$\mathrm{Var}(U)=\underbrace{\Sigma\otimes I_T}_{MT\times MT}=\begin{bmatrix}\begin{bmatrix}\sigma_{11} & & \\ & \ddots & \\ & & \sigma_{11}\end{bmatrix} & \begin{bmatrix}\sigma_{12} & & \\ & \ddots & \\ & & \sigma_{12}\end{bmatrix} & & \\ & & \ddots & \\ & & & \begin{bmatrix}\sigma_{MM} & & \\ & \ddots & \\ & & \sigma_{MM}\end{bmatrix}\end{bmatrix}$$

$E[V]=\underbrace{E[U]}_{=0}B^{-1}=0$

$\mathrm{Var}(v_t)=(B^{-1})'\Sigma B^{-1}=\Omega$

$\mathrm{Var}(V)=\Omega\otimes I_T$

- $E[Z'U]=0$ by assumption
- B is nonsingular
- Z has full column rank $\Rightarrow \frac{Z'Z}{T}\xrightarrow{p}M$, and we want M to be nonsingular
- Joint normality of U $\quad U\sim N(\ )$ matrix-variate normal

$$\text{plim } \frac{Y'u}{T} = \text{plim } \frac{1}{T}(Z\Pi + V)'u = \text{plim } \frac{\Pi' Z' u}{T} + \text{plim } \frac{V'u}{T}$$

$$= \text{plim } \frac{(B^{-1})' u'u}{T} = \underbrace{(B^{-1})'}_{\text{fiddle with this}} \underbrace{\Sigma}_{\text{or with this.}} \neq 0$$

If B is triangular and Σ is diagonal, this is referred to as a "recursive system."

- Will burn in hell if you assume a recursive system.

Identification problem:

$(\Pi, \Omega) \rightarrow (B, \Gamma, \Sigma)$ uniquely

need restrictions. $\Pi = -\Gamma B^{-1}$ $\Omega = (B^{-1})' \Sigma B^{-1}$

Linear restrictions: $\underbrace{[G \quad H]}_{\Phi} \begin{bmatrix} B \\ \Gamma \end{bmatrix} = \varphi$

- includes exclusions
- includes normalizations

For equation j, have g_j linear restrictions

$$\begin{array}{cc} & \begin{array}{cc} M & K \end{array} \\ \begin{array}{c} K \\ g_j \end{array} & \begin{bmatrix} \Pi & I_K \\ \multicolumn{2}{c}{\Phi_j} \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{bmatrix} B_j \\ \Gamma_j \end{bmatrix} \begin{array}{c} M \\ K \end{array} \end{array} = \begin{bmatrix} 0 \\ \varphi_j \end{bmatrix} \begin{array}{c} K \\ g_j \end{array}$$

$$\Pi B + \Gamma = 0 \Rightarrow \Pi = -\Gamma B^{-1}$$

$$G B + H \Gamma = \varphi \iff \Phi \begin{bmatrix} B \\ \Gamma \end{bmatrix} = \varphi$$

Exclusion restrictions: $\Phi_{jk} = [0 \cdots 0 1 0 \cdots 0]$

Normalizations:

Conditions

1] # restrictions including normalizations $\geq$ # endogenous variables.
(order condition)

2] rank $\begin{bmatrix} \Pi & I_K \\ \Phi_j & \end{bmatrix} = M+K \iff \text{rank } \Phi_j \begin{bmatrix} B \\ \Gamma \end{bmatrix} = M$

$\Rightarrow$ equation j is identified.

---

1. Consider the following system of equations:

$$\begin{cases} y_1 = \beta_{12} y_2 + \gamma_{11} z_1 + \gamma_{12} z_2 + u_1 \\ y_2 = \beta_{21} y_1 + \gamma_{23} z_3 + u_2 \end{cases}$$

1] Rewrite the system in matrix notation
2] Construct the matrix of constraints $\Phi_j \ \forall j$
3] Check order and rank

1] $$[y_1 \ y_2] \underbrace{\begin{bmatrix} 1 & -\beta_{21} \\ -\beta_{12} & 1 \end{bmatrix}}_{B} + [z_1 \ z_2 \ z_3] \underbrace{\begin{bmatrix} -\gamma_{11} & 0 \\ -\gamma_{12} & 0 \\ 0 & -\gamma_{23} \end{bmatrix}}_{\Gamma} = [u_1 \ u_2]$$

$$[y_1 \ y_2 \ z_1 \ z_2 \ z_3] \underbrace{\begin{bmatrix} 1 & -\beta_{21} \\ -\beta_{12} & 1 \\ -\gamma_{11} & 0 \\ -\gamma_{12} & 0 \\ 0 & -\gamma_{23} \end{bmatrix}}_{A} = [u_1 \ u_2] \qquad A = \begin{bmatrix} B \\ \hline \Gamma \end{bmatrix}$$

2] eqn 1: $\Phi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

eqn 2: $\Phi_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$.

3] $\text{rank } [\Phi_1 A] = rk\begin{bmatrix} 1 & -\beta_{21} \\ 0 & -\gamma_{23} \end{bmatrix} = 2$ if $\gamma_{23} \neq 0$

$\text{rank } [\Phi_2 A] = rk\begin{bmatrix} -\beta_{21} & 1 \\ -\gamma_{11} & 0 \\ -\gamma_{12} & 0 \end{bmatrix} = 2$ if either $\gamma_{12} \neq 0$ or $\gamma_{11} \neq 0$.

$$y_1 = \beta_{12} y_2 + \gamma_{11} z_1 + u_1$$

$$y_2 = \beta_{21} y_1 + \gamma_{21} z_1 + \gamma_{22} z_2 + \gamma_{23} z_3 + u_2$$

$$A = \begin{bmatrix} 1 & -\beta_{21} \\ -\beta_{12} & 1 \\ -\gamma_{11} & -\gamma_{21} \\ 0 & -\gamma_{22} \\ 0 & -\gamma_{23} \end{bmatrix} \qquad \Phi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \Phi_2 = [0\ 1\ 0\ 0\ 0]$$

$$\Phi_1 A = \begin{bmatrix} 1 & -\beta_{21} \\ 0 & -\gamma_{22} \\ 0 & -\gamma_{23} \end{bmatrix}$$

$\text{rank } \Phi_1 A = 2$ if $\gamma_{22} \neq 0$ or $\gamma_{23} \neq 0$

eqn 2 is not identified since it fails the order conditions.

$$\Pi = \begin{bmatrix} -\gamma_{11} & -\gamma_{21} \\ 0 & -\gamma_{22} \\ 0 & -\gamma_{23} \end{bmatrix} \frac{-1}{1-\beta_{21}\beta_{12}} \begin{bmatrix} 1 & -\beta_{21} \\ -\beta_{12} & 1 \end{bmatrix}$$

$$= \frac{-1}{1-\beta_{21}\beta_{12}} \begin{bmatrix} -\gamma_{11} + \gamma_{21}\beta_{12} & \gamma_{11}\beta_{21} - \gamma_{21} \\ \gamma_{22}\beta_{12} & \gamma_{22} \\ \gamma_{23}\beta_{12} & \gamma_{23} \end{bmatrix} = \begin{bmatrix} \Pi_{11} & \Pi_{21} \\ \Pi_{12} & \Pi_{22} \\ \Pi_{13} & \Pi_{23} \end{bmatrix}$$

$$\Rightarrow \frac{\Pi_{12}}{\Pi_{22}} = \beta_{12} \quad \text{and} \quad \frac{\Pi_{13}}{\Pi_{23}} = \beta_{13}$$

Identification based on reduced form:

Partition $B_j = \begin{bmatrix} \tilde{B}_j \\ \tilde{\tilde{B}}_j \end{bmatrix} \begin{matrix} \updownarrow r_1 & \text{included endo. variables in eq. } j \\ \updownarrow n-r_1 & \text{excluded endo. variables in eq. } j. \end{matrix}$

$\Gamma_j = \begin{bmatrix} \tilde{\Gamma}_j \\ \tilde{\tilde{\Gamma}}_j \end{bmatrix} \begin{matrix} \updownarrow s_1 & \text{included exog. variables in } j. \\ \updownarrow K-s_1 & \text{excluded exog. variables in } j. \end{matrix}$

$$\underbrace{\begin{array}{c|cc} & r_1 & n-r_1 \\ \hline s_1 & \Pi_{11} & \Pi_{12} \\ K-s_1 & \Pi_{21} & \Pi_{22} \end{array}}_{K \times M} \underbrace{\begin{bmatrix} \tilde{B}_j \\ \tilde{\tilde{B}}_j \end{bmatrix}}_{M \times 1} = \underbrace{\begin{bmatrix} \tilde{\Gamma}_j \\ \tilde{\tilde{\Gamma}}_j \end{bmatrix}}_{K \times 1} \begin{matrix} s_1 \\ K-s_1 \end{matrix}$$

order cond: $K - s_1 \geq r_1 - 1$

\# excluded exog $\geq$ # incl. endo. $-$ normalization

rank cond: $\text{rank } \Pi_{21} = r_1 - 1$