Today: ∘ more math
∘ IV from 382
∘ Weak instruments

Read Jerry's handbook chapter (2/3rds of class)

Kronecker products

$\underset{p\times q}{A}, \underset{r\times s}{B} \Rightarrow A \otimes C = \underset{pr\times qs}{C}$

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1q}B \\ \vdots & & \\ a_{p1}B & \cdots & a_{pq}B \end{bmatrix}$$

Vec operator $A(:)$

$$\underset{p\times q}{A} = \left[a_1 \,\middle|\, a_2 \,\middle|\, \cdots \,\middle|\, a_q \right]$$

$$\underset{pq\times 1}{\text{vec } A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_q \end{bmatrix}$$

$$(A \otimes B) \text{ vec } x = \text{vec}(B x A^T)$$

$$\bigotimes_{i=1}^{R} X_i = X_1 \otimes X_2 \otimes \cdots \otimes X_R$$

$$\Rightarrow X^{\otimes R} = \bigotimes_{i=1}^{R} X$$

Matrix derivatives

Let the elements of $\underset{s\times t}{Y}$ be functions of the elements of $\underset{m\times n}{X}$ and let the elements of X be functionally independent $\frac{\partial x_{ij}}{\partial x_{ek}} = 0 \quad (i,j) \neq (e,k)$

Let $\frac{d}{dX} = \left[\partial / \partial x_{ij} \right]$ be a matrix of derivative operators. Then the derivative of Y wrt X is an $ms \times nt$ matrix given by:

$$\frac{dY}{dX} = Y \otimes \frac{d}{dX}.$$ (Magnus and Klendecker is a reference)

Will get a matrix consisting of partial derivatives of all elements of X wrt all elements of Y

$$\frac{d(YZ)}{dX} = \left(\frac{dY}{dX}\right)(Z \otimes I_n) + (Y \otimes I_n)\left(\frac{dZ}{dX}\right)$$

$\underset{1\times 1}{f(\beta)} = \exp\{X'\beta\}$ where $\underset{1\times T}{X'} = [X_1, \dots, X_T]$, $\beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_T \end{bmatrix}$

$$\underbrace{\frac{d}{d\beta} f(\beta)}_{T\times 1} = \begin{bmatrix} X_1 \exp\{X'\beta\} \\ \vdots \\ X_T \exp\{X'\beta\} \end{bmatrix} = X \exp\{X'\beta\}$$

$$\frac{d^2}{d\beta^2} f(\beta) = \left(\frac{dX}{d\beta}\right)(\exp\{X'\beta\} \otimes I_T) + (X \otimes I_T)\frac{d}{d\beta}\exp\{X'\beta\}$$
$$= (X \otimes X)\exp\{X'\beta\} = X^{\otimes 2}\exp\{X'\beta\}$$

Recall from 382

$\underset{T\times L}{Z}$ - orthogonal to ε but related to $\underset{T\times K}{X}$

- $L \geq K$

$Y = \beta X + \varepsilon$ allow for correlations between ε and v.
$X = \Pi Z + v$

$\underset{T\times K}{W} = \underset{T\times L}{Z}\ \underset{L\times K}{\hat{A}}$ → linear combination

$$\hat{\beta}_{IV} = (W'X)^{-1}W'Y$$
$$= (\hat{A}'Z'X)^{-1}\hat{A}'Z'Y$$

2SLS: $W = Z\hat{A} = Z\underbrace{(Z'Z)^{-1}Z'X}_{\hat{A}}$

$$\Rightarrow \hat{\beta}_{2SLS} = (X'P_Z X)^{-1}X'P_Z Y$$

This is the result of
$$\min_{\beta}\ (Y - X\beta)'P_Z(Y - X\beta)$$

OIV: $W = Z\hat{A} = Z\hat{V}^{-1}Z'X$

where $\frac{1}{T}\hat{V} \xrightarrow{p} V = \text{plim}\,\frac{1}{T}E[Z'\varepsilon\varepsilon'Z] = \text{plim}\,\frac{1}{T}E[Z'\Omega Z]$

If $\underbrace{\hat{V} = Z'Z}_{\text{homoskedasticity}} \Rightarrow \hat{\beta}_{2SLS} = \hat{\beta}_{OIV}$

What happens if $K=L$?

- $\hat{\beta}_{OIV} \overset{A}{\sim} \hat{\beta}_{2SLS}$

$\hat{\beta}_{OIV} = (W'X)^{-1}W'Y \qquad W = Z\hat{V}^{-1}Z'X$

$= (X'Z\hat{V}^{-1}Z'X)^{-1}X'Z\hat{V}^{-1}Z'(X\beta+\varepsilon)$

$= \beta + (X'Z\hat{V}^{-1}Z'X)^{-1}X'Z\hat{V}^{-1}Z'\varepsilon$

Consistency:

$$\text{plim}(\hat{\beta}_{OIV}-\beta) = \text{plim}\,(\underbrace{X'Z}_{Q'_{ZX}}\underbrace{\hat{V}^{-1}}_{V^{-1}}\underbrace{Z'X}_{Q_{ZX}})^{-1}\underbrace{X'Z}_{Q'_{ZX}}\underbrace{\hat{V}^{-1}}_{V^{-1}}\underbrace{Z'\varepsilon}_{0}$$

$= 0$

asymptotic normality:

$$\sqrt{T}(\hat{\beta}_{OIV}-\beta) = (\underbrace{\frac{X'Z}{T}}_{Q'_{ZX}}\underbrace{\hat{V}^{-1}}_{V^{-1}}\underbrace{\frac{Z'X}{T}}_{Q_{ZX}})^{-1}\underbrace{\frac{X'Z}{T}}_{Q'_{ZX}}\underbrace{\hat{V}^{-1}}_{V^{-1}}\underbrace{\frac{Z'\varepsilon}{\sqrt{T}}}_{\xrightarrow{d} N(0,V)}$$

$\xrightarrow{d} N(0, (Q'_{ZX}V^{-1}Q_{ZX})^{-1})$

For 2SLS,

$$\sqrt{T}(\hat{\beta}_{2SLS}-\beta) \xrightarrow{d} N(0,\Omega)$$

$$\Omega = (Q_{ZX}' Q_{ZZ}^{-1} Q_{ZX})^{-1} Q_{ZX}' Q_{ZZ}^{-1} V Q_{ZZ}^{-1} Q_{ZX} (Q_{ZX}' Q_{ZZ}^{-1} Q_{ZX})^{-1}$$

<u>Weak instruments</u>

<u>always</u> run a test for weak instruments.

$$\underset{1\times1}{Y_t} = \underset{1\times1}{X_t}\,\beta + u_t \qquad u_t \overset{iid}{\sim} N(0,\sigma_u^2) \qquad X_t \text{ endogenous}$$

$$E[X_t u_t] \neq 0$$

Suppose $\exists\ Z_t$ s.t. $\begin{bmatrix} Z_t \\ X_t \end{bmatrix} \overset{iid}{\sim} N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_{ZZ} & \sigma_{ZX} \\ \sigma_{ZX} & \sigma_{XX} \end{bmatrix}\right)$

$$\hat{\beta}_{IV} - \beta_0 = \left(\frac{Z'X}{T}\right)^{-1}\left(\frac{Z'u}{T}\right)$$

$$= \left(\frac{\frac{1}{T}Z'X}{\hat{\sigma}_{ZZ}\,\hat{\sigma}_{XX}}\right)^{-1}\left(\frac{\frac{1}{T}Z'u}{\hat{\sigma}_{ZZ}\,\hat{\sigma}_u}\right)\frac{\hat{\sigma}_u}{\sigma_{XX}}$$

$$= \frac{\hat{\rho}_{Zu}}{\hat{\rho}_{ZX}}\,\frac{\hat{\sigma}_u}{\hat{\sigma}_{XX}}$$

First order asymptotics:

$$\sqrt{T}(\hat{\beta}_{IV}-\beta_0) = \underset{\substack{\downarrow p \\ \text{const}}}{\hat{\rho}_{ZX}^{-1}}\ \underset{\substack{\downarrow p \\ \text{const}}}{\frac{\hat{\sigma}_u}{\hat{\sigma}_{XX}}}\ \underset{\substack{\downarrow d \\ N(\cdot,\cdot)}}{\left(\frac{\hat{\rho}_{Zu}}{\sqrt{T}}\right)} \xrightarrow{d} N\left(0,\frac{\sigma_u^2}{\sigma_{XX}^2\,\rho_{ZX}^2}\right)$$

What if $\hat{\rho}_{ZX}^{-1} \to v$ where v is a random variable

i.e. $\hat{\rho}_{ZX} \to N(0,1)$

$$\sqrt{T}(\hat{\beta}_{IV}-\beta_0) \text{ ``}\to\text{'' } \frac{N(0,1)}{N(0,1)} \frac{\sigma_u}{\sigma_{XX}} \text{ ``=''} \text{ Cauchy}$$

The variance should "blow up." Weak instruments manifest themselves as bias. Cannot detect weak instruments just by looking at standard errors.

Concentration Parameter: $\mu^2 = \dfrac{\Pi' Z' Z \Pi}{\sigma_{vv}}$