14.383

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Handout # 7

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· Finite-sample issues of 2SLS:

· Model: $\underset{(T\times1)}{y_1} = \underset{(T\times1)}{y_2}\underset{(1\times1)}{\beta} + \underset{(T\times1)}{\varepsilon_1}$: equation 1 has 1 endog. var. on the RHS (for simple (we can also have RHS exog. vars., but here we've partialled them out!).

$\underset{(T\times1)}{y_2} = \underset{(T\times K)(K\times1)}{Z\Pi} + \underset{(T\times1)}{v_2}$: equation for y_2 (the endog. regressor in equ. in RF.

Assume: $$\begin{cases} E[\varepsilon_1] = E[v_2] = E[Z'\varepsilon_1] = E[Z'v_2] = \\ E[\varepsilon_1\varepsilon_1'] = \sigma_\varepsilon^2 I_T \\ E[v_2 v_2'] = \sigma_v^2 I_T \\ E[\varepsilon_1 v_2'] = \sigma_{\varepsilon v} I_T \end{cases}$$

· We know 2SLS on equ. 1 using Z as instruments is consistent

$$\hat\beta_{2SLS} = \left(\frac{\frac{1}{T}\sum \hat y_2' y_1}{\frac{1}{T}\sum \hat y_2'\hat y_2}\right) (\hat y_2'\hat y_2)^{-1}\hat y_2' y_1 = (y_2' P_Z y_2)^{-1} y_2' P_Z y_1$$

$$\Rightarrow \operatorname{plim}(\hat\beta_{2SLS}-\beta) = \operatorname{plim}\left\{\left[\underbrace{\left(\frac{y_2'Z}{T}\right)}_{Q_{Y2}'}\underbrace{\left(\frac{Z'Z}{T}\right)^{-1}}_{Q_{ZZ}^{-1}}\underbrace{\left(\frac{Z'y_2}{T}\right)}_{Q_{Y2}}\right]^{-1}\underbrace{\left(\frac{y_2'Z}{T}\right)}_{Q_{Y2}'}\underbrace{\left(\frac{Z'Z}{T}\right)^{-1}}_{Q_{ZZ}^{-1}}\underbrace{\left(\frac{Z'\varepsilon_1}{T}\right)}_{0}\right\} = 0.$$

But in finite samples it's biased:

$$\Rightarrow E[\hat\beta_{2SLS}-\beta] = E\{(y_2'P_Z y_2)^{-1} y_2' P_Z\varepsilon_1\} = E\{(\hat y_2'\hat y_2)^{-1}(Z\Pi + v_2)' P_Z \varepsilon_1\} =$$

$$= E\{(\hat y_2'\hat y_2)^{-1}\Pi' Z'\cancel{\overbrace{Z(Z'Z)^{-1}Z'}^{P_Z}}\varepsilon_1\} + E\{(\hat y_2'\hat y_2)^{-1} v_2' P_Z \varepsilon_1\}$$

$$= 0 + E\{(\hat y_2'\hat y_2)^{-1} v_2' P_Z\varepsilon_1\}$$

$$\approx \frac{E[v_2'P_Z\varepsilon_1]}{E[\hat y_2'\hat y_2]} = \frac{K\,\sigma_{\varepsilon v}}{T\cdot R_{FS}^2\cdot Var(y_2)} = \frac{K\,\sigma_{\varepsilon v}}{T\Pi' M\Pi + K\sigma_v^2}$$

(inverse of) F-statistic of First Stage $= \dfrac{\sigma_{\varepsilon v}}{\left(\frac{T}{K}\right)\Pi' M\Pi + \sigma_v^2}$

(because:

$$\begin{aligned} E[v_2'P_Z\varepsilon_1] &= tr\,E[v_2'P_Z\varepsilon_1] = \quad \text{it's a scalar!} \\ &= E[tr[v_2'P_Z\varepsilon_1]] = \\ &= E[tr[\varepsilon_1 v_2' P_Z]] = \quad \text{it's a linear operator!} \\ &= tr[E[\varepsilon_1 v_2'] P_Z] = \\ &= \sigma_{\varepsilon v}\cdot rank(P_Z) = \quad \text{it's idempotent.} \\ &= \sigma_{\varepsilon v}\cdot K \end{aligned}$$

R^2 of the First-Stage regression!

$$E[\hat y_2'\hat y_2] = E\left[\left(\frac{\hat y_2'\hat y_2}{y_2' y_2}\right) T\left(\frac{y_2'y_2}{T}\right)\right] \approx T\cdot R_{FS}^2\cdot Var(y_2) \quad \text{— if we have de-meaned data!}$$

$$= E[y_2' P_Z y_2] = E[(Z\Pi + v_2)' P_Z (Z\Pi + v_2)] =$$

$$= T\cdot\Pi' E\left(\frac{Z'Z}{T}\right)\Pi + E[v_2'P_Z v_2] + \underbrace{2\Pi' E[Z'v_2]}_{0} =$$

$$= T\cdot\Pi' M\Pi + \sigma_v^2 K \quad \text{(similar as above)}.$$

- We can also compute the OLS bias:

$$\hat{\beta}_{OLS} = \left(\frac{\frac{1}{T}\sum y_2' y_1}{\frac{1}{T}\sum y_2' y_2}\right) (y_2'y_2)^{-1} y_2' y_1$$

$$\Rightarrow E[\hat{\beta}_{OLS} - \beta] = E\left[(y_2'y_2)^{-1} y_2' \varepsilon_1\right] = E\left[(y_2'y_2)^{-1}(z\pi + v_2)'\varepsilon_1\right] =$$

$$= \cancel{E\left[(y_2'y_2)^{-1}\pi' z'\varepsilon_1\right]} + E\left[(y_2'y_2)^{-1} v_2'\varepsilon_1\right] =$$

$$\approx \frac{E[v_2'\varepsilon_1]}{E[y_2'y_2]} = \frac{T\sigma_{\varepsilon v}}{T\cdot\pi' M\pi + T\cdot\sigma_v^2} = \frac{\sigma_{\varepsilon v}}{\pi' M \pi + \sigma_v^2}$$

(because: $E[y_2'y_2] = E[(z\pi + v_2)'(z\pi + v_2)] =$

$$= T\cdot \pi' E\left(\frac{z'z}{T}\right)\pi + E[v_2'v_2] + \cancel{2\pi' \underbrace{E[z'v_2]}_{0}} =$$

$$= T\cdot\pi' M\pi + T\cdot\sigma_v^2$$

).

- Important things to note:
  - The OLS bias doesn't depend on T (= the number of observations), so it does not go away as $T \to \infty$, which makes sense, since OLS is not only biased but inconsistent too!
  - The 2SLS bias does go away as $T \to \infty$. 2SLS is consistent!
    Intuition for the 2SLS bias: the average covariance between $P_z\varepsilon_1$ and v_2 is asymptotically negligible but has a non-negligible expectation in any finite sample.
    The 2SLS bias gets worse...
    ... the higher the number of instruments we use (K).
    ... the smaller the sample size (T).
    ... the smaller the R^2 of the first stage (a measure of correlation between the instruments, Z, and the "instrumentee", y_2).
    ... the bigger the correlation between the unobserved determinants of y_1 and the unobserved determinants of the endogenous regressor y_2 (ie: $\sigma_{\varepsilon v}$).
  - In general, if $|\pi| \gg 0$, we have that $\text{bias}(\hat{\beta}_{2SLS}) < \text{bias}(\hat{\beta}_{OLS})$ as long as $\left(\frac{T}{K}\right) > 1$ (which is usually the case!).

~~But as we approach the situation where~~
But the bias of 2SLS gets closer to that of OLS as the correlation between y_2 and Z gets weaker. In the limit, we have the no-identification case where $\pi = 0$, we have that $\text{bias}(\hat{\beta}_{2SLS}) = \text{bias}(\hat{\beta}_{OLS}) = \frac{\sigma^2_{\varepsilon v}}{\sigma_v^2}$, and it does not then depend on sample size, because in that case we cannot really ~~get a consistent estimator of β~~ get a consistent estimator of β!!

the "weak instruments problem"

Notes on Bias in Estimators for Simultaneous Equation Models

Jinyong Hahn and Jerry Hausman[1]

June, 2001

We begin with the simplest model specification with one right hand side (RHS) jointly endogenous variable so that the left hand side variable (LHS) depends only on the single jointly endogenous RHS variable. This model specification accounts for other RHS predetermined (or exogenous) variables, which have been "partialled out" of the specification.[2] We will assume that

$$(1) \qquad y_1 = \beta y_2 + \varepsilon_1 = \beta z \pi_2 + v_1$$

$$(2) \qquad y_2 = z\pi_2 + v_2,$$

where $\dim(\pi_2) = K$. Thus, the matrix z is the matrix of all predetermined variables, and equation (2) is the reduced form equation for y_2 with coefficient vector π_2. We also assume homoscedasticity:

$$(3) \qquad \begin{pmatrix} v_{1i} \\ v_{2i} \end{pmatrix} \sim N(0,\Omega) \sim N\left(0, \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix}\right).$$

We use the following notation:

$$y \equiv \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad z \equiv \begin{pmatrix} z_1' \\ \vdots \\ z_n' \end{pmatrix}, \quad \sigma_\varepsilon^2 \equiv \mathrm{Var}(\varepsilon_{1i}), \quad \sigma_{\varepsilon v_2} \equiv \mathrm{Cov}(\varepsilon_{1i}, v_{2i}), \quad \sigma_{\varepsilon v_1} \equiv \mathrm{Cov}(\varepsilon_{1i}, v_{1i}).$$

I. Bias in 2SLS and OLS

A common finding in empirical research is that when 2SLS is used the coefficient estimate increases in magnitude from the OLS estimate. However, in finite samples under certain situations even when 2SLS is used on equation (1), bias remains because an

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estimate of π_2 from equation (2) is used, since the true parameters are unknown. We now demonstrate how this result occurs.

Suppose that $z\pi_2$ is measured without error. Then, OLS of y_1 on $z\pi_2$ would be unbiased. Instead, $z\pi_2$ must be estimated, *i.e.*, we have to rely on 2SLS. Let $\hat{\pi}_2$ denote the first stage OLS estimator. We have

$$(4) \qquad b_{2SLS} - \beta = \frac{\sum_{i=1}^{n} \left(v_{1i} - \beta z_i' \cdot (\hat{\pi}_2 - \pi_2)\right) \cdot z_i'\hat{\pi}_2}{\sum_{i=1}^{n} (z_i'\hat{\pi}_2)^2} = \frac{\sum_{i=1}^{n} \left(v_{1i} - \beta z_i' \cdot (\hat{\pi}_2 - \pi_2)\right) \cdot z_i'\hat{\pi}_2}{R_f^2 \cdot \sum_{i=1}^{n} y_{2i}^2},$$

where R_f^2 is the R^2 in the first stage regression to obtain $\hat{\pi}_2$. It can be shown that:

$$E\left[\sum_{i=1}^{n} \left(v_{1i} - \beta z_i' \cdot (\hat{\pi}_2 - \pi_2)\right) \cdot z_i'\hat{\pi}_2\right] = K \cdot \sigma_{\varepsilon v_2}, \text{ and } E\left[\sum_{i=1}^{n} (z_i'\hat{\pi}_2)^2\right] = n \cdot \pi_2' R \pi_2 + K \cdot \sigma_{v_2 v_2},$$

where $R \equiv E[z'z/n]$. Here, $K \cdot \sigma_{v_2 v_2}$ is the expectation of the term $v_2' P_z v_2$, which is $\sigma_{v_2 v_2}$ times a χ^2- random variable with expectation equal to the dimension of the projection matrix P_z.

Therefore, we expect bias approximately equal to

$$(5) \qquad E[b_{2SLS}] - \beta \approx \frac{K \cdot \sigma_{\varepsilon v_2}}{R_f^2} \frac{1}{\sum_{i=1}^{n} y_{2i}^2} \approx \frac{K \sigma_{\varepsilon v_2}}{n \cdot \pi_2' R \pi_2 + K \cdot \sigma_{v_2 v_2}}$$

Equation (5) indicates that the bias is monotonically increasing in $\sigma_{\varepsilon v_2}$ and K, but monotonically decreasing in R_f^2. Conventional asymptotics, which lets $n \to \infty$ keeping DGP fixed, ignores the influence of $\sigma_{\varepsilon v_2}$, K, and R_f^2.

For comparison purposes we calculate the bias of OLS. We find approximately that

$$(6) \qquad E[b_{OLS}] - \beta \approx \frac{\mathrm{Cov}(y_2, \varepsilon)}{\mathrm{Var}(y_2)} \approx \frac{\sigma_{\varepsilon v_2}}{\pi_2' R \pi_2 + \sigma_{v_2 v_2}}$$

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[2] See Hahn and Hausman (1999), available at http://web.mit.edu/jhausman/www/, for more complete

Note that in equation (5) the denominator typically becomes large as the sample size n becomes large so that the bias of 2SLS decreases. However, in the OLS bias equation (6) the denominator does not change size as n increases so that the bias does not decrease. Thus, 2SLS is consistent and OLS is inconsistent, as is well known.

II. No Identification

We now use equation (5) to explore what happens in the unidentified situation of $\pi_2 = 0$. The denominator of equation (5) becomes $K \cdot \sigma_{v_2 v_2}$. Thus, when $\pi_2 = 0$, equation (5) predicts the bias of the 2SLS estimator to be approximately

$$(7) \qquad E[b_{2SLS}] - \beta \approx \frac{\sigma_{\varepsilon v_2}}{\sigma_{v_2 v_2}}$$

In large samples the result holds in the limit without the necessity of assuming that the stochastic disturbances are normal. Note that the bias does not decrease here as n becomes large as it did in the last section. This result is expected because without identification we cannot find a consistent estimator of beta.

We now compare this 2SLS bias with the bias of OLS on equation (1) again where no identification exists so that $\pi_2 = 0$. We use equation (6). When $\pi_2 = 0$, the denominator is equal to $\sigma_{v_2 v_2}$. Thus, we find that the bias of OLS is the same as the bias of 2SLS in the unidentified case of $\pi_2 = 0$:

$$(8) \qquad E[b_{OLS}] - \beta \approx \frac{\sigma_{\varepsilon v_2}}{\sigma_{v_2 v_2}}$$

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discussion of models with other RHS predetermined variables.

See Phillips (1989) for related results.

III. Local Non-Identification

We now consider what happens when we are close to being unidentified so that $\pi_2 = a/\sqrt{n}$, where the vector a has dimension *K*. Thus, the reduced form coefficients are "local to zero". Stock and Staiger (1997) refer to this situation as "weak instruments". We disagree somewhat with this terminology because the result of badly biased IV estimators also depends on the value of covariance term in the numerator of equation (5) as we discuss in Hahn and Hausman (1999).

With $\pi_2 = a/\sqrt{n}$, equation (5) predicts the bias of 2SLS to be

$$(9) \qquad E[b_{2SLS}] - \beta \approx \frac{K\sigma_{\varepsilon v_2}}{a'Ra + K\sigma_{v_2 v_2}} = \frac{\sigma_{\varepsilon v_2}}{\frac{1}{K}a'Ra + \sigma_{v_2 v_2}}$$

Equation (9) is an approximation to the asymptotic bias of 2SLS under the asymptotics where $\pi_2 = a/\sqrt{n}$. When *K* is sufficiently large, the difference between equation (9) and the asymptotic bias is negligible. See Chao and Swanson (2000, Theorem 3.1 (c)). On the other hand, equation (6) predicts the approximate bias for OLS to be:

$$(10) \qquad E[b_{OLS}] - \beta \approx \frac{\sigma_{\varepsilon v_2}}{\frac{1}{n}a'Ra + \sigma_{v_2 v_2}}$$

Comparison of (9) and (10) suggest that the bias of 2SLS is smaller than OLS as long as *K*<*n*, a condition which will always be satisfied in practice.

We have considered three asymptotic approximation: (i) $\pi_2 \neq 0$ and fixed; (ii) $\pi_2 = a/\sqrt{n}$, $a \neq 0$; (iii) $\pi_2 = 0$. For the first two cases, our approximate bias formulae predict that 2SLS has less bias than OLS. For the last case, our formulae predict that 2SLS has approximately equal bias as OLS.

IV. Bias Corrected 2SLS

We can also use equation (5) to construct an approximately unbiased 2SLS estimator. While it first appears that we have only one equation (moment) and two unknowns in β and $\sigma_{\varepsilon v_2}$, it turns out that this second parameter is a function of beta:

$$\sigma_{\varepsilon v_2} = E\left[\frac{1}{N-K}(y_2'Q_z)(y_1 - y_2\beta)\right] \quad \text{where} \quad Q_z \equiv I - P_z.$$

Now we can solve for β which is a linear equation. The derivation is:

$$E[\hat{\beta} - \beta] \approx \frac{K\sigma_{\varepsilon v_2}}{y_2' P_z y_2} \approx E\left[\frac{M}{N-K}(y_2'Q_z)(y_1 - y_2\beta)\right] = E[dq'(y_1 - y_2\beta)]$$

where $M = K / y_2'P_z y_2$, $d = M/(N-K)$ and $q' = y_2'Q_z$. Thus we can solve for beta to find a bias corrected estimator β_{BC}:

$$\hat{\beta}_{BC} = (\hat{\beta} - dq'y_1)/(1 - dq'y_2)$$

If we now consider the (approximate) bias of the estimator we find it to be zero by construction. Thus, the estimator is approximately unbiased as claimed.

This estimator turns out to be the same as Nagar's estimator (EMA 1959), which was derived in a considerably more difficult manner using a higher order expansion approach. This equivalence can be seen from:

$$\hat{\beta}_{BC} = \hat{\beta}_N = \frac{\dfrac{y_2'P_z y_1}{y_2'P_z y_2} - \dfrac{K}{N-K}\dfrac{y_1'Q_z y_2}{y_2'P_z y_2}}{1 - \dfrac{K}{N-K}\dfrac{y_2'Q_z y_2}{y_2'P_z y_2}} = \frac{y_2'P_z y_1 - \dfrac{K}{N-K}y_1'Q_z y_2}{y_2'P_z y_2 - \dfrac{K}{N-K}y_2'Q_z y_2}$$

Unfortunately, the estimator has no moments, and performs poorly when the model is nearly non-identified.[3] This poor performance follows by noting that the denominator is zero when $\pi_2 = 0$. The Nagar estimator "blows up" in this situation in contrast to the 2SLS estimator, which is inconsistent but has its moments existing. For near non-identification, the Nagar estimator similarly works poorly because the non-existence of moments from the denominator being near zero leads to poor results in many situations. Hahn, Hausman, and Kuersteiner (2001) give Monte Carlo results that demonstrate the poor performance of the Nagar estimator in this situation. Thus, the Nagar estimator is not very useful in the situation where 2SLS has substantial bias. Hahn, Hausman, and Kuersteiner (2001) explore alternative estimators to use in this situation.

[3] See Sawa (1972).

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