14.383

Fall 2003

Handout #5

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FIML

Recall from ~~last Section~~ handout #4 that the FOC's for FIML reduce to:

$$\boxed{\Pi_j'(Z'U)\,\Sigma_{(j)}^{-1} = 0} \quad (1_j)$$

$$\boxed{Z_j'U\,\Sigma_{(j)}^{-1} = 0} \quad (2_j)$$

where
- $\Sigma_{(j)}^{-1}$ is the j^{th} column of Σ^{-1}, $u_i' \sim N(0, \Sigma)$
- Z_j is the matrix of exogenous variables that appears in Equation j, i.e. $y_j = Y_j\beta_j + Z_j\gamma_j + u_j$
- Π_j is the matrix of reduced form parameters for Y_j, the set of endogenous variables that appear on the RHS of equation j.

Noting that $Z_j = Z C_j$ and defining $D_j = [\Pi_j \;\; C_j]$, we have, grouping (1_j), (2_j),

$$\boxed{D_j'(Z'U)\,\Sigma_{(j)}^{-1} = 0}$$

FIML

We saw last time, grouping the equations for the j equations,

$$\underbrace{\begin{bmatrix} D_1' & & & 0 \\ & D_2' & & \\ & & \ddots & \\ 0 & & & D_m' \end{bmatrix}}_{\bar{D}} \begin{bmatrix} Z'U\Sigma_{(1)}^{-1} \\ \vdots \\ Z'U\Sigma_{(m)}^{-1} \end{bmatrix} = 0$$

and using $Vec(ABC) = (C' \otimes A)\, Vec(B)$,

this gives

$$\boxed{\bar{D}\,(\Sigma^{-1} \otimes Z')(\bar{X}\bar{\delta} - \bar{y}) = 0}$$

Solving this equation for $\bar{\delta}$ gives the FIML estimator:

$$\hat{\bar{\delta}}_{FIML} = \left[\bar{D}\,(\Sigma^{-1} \otimes Z')\,\bar{X}\right]^{-1} \left[\bar{D}\,(\Sigma^{-1} \otimes Z')\,\bar{y}\right]$$

In computing this estimator, an estimate of $\bar{D}$, Σ^{-1} would need to be plugged in.

Diagonal elements of $\bar{D}$ (e.g. D_j) contain submatrices of Π (e.g. $D_j = [\Pi_j \;\; C_j]$).

Π can be consistently estimated by OLS on the reduced form. Σ^{-1} can be estimated as in 3SLS.

Comparing FIML, 3SLS

$$\hat{\bar{\delta}}_{3SLS} = (\bar{X}'(\Sigma^{-1}\otimes P_z)\bar{X})^{-1}(\bar{X}'(\Sigma^{-1}\otimes P_z)\bar{y})$$

$$\Rightarrow \quad 0 = (\bar{X}'(\Sigma^{-1}\otimes P_z)\bar{X})\,\hat{\bar{\delta}}_{3SLS} - \bar{X}(\Sigma^{-1}\otimes P_z)\bar{y}$$

$$0 = \bar{X}'(\Sigma^{-1}\otimes P_z)(\bar{X}\,\hat{\bar{\delta}}_{3SLS} - \bar{y})$$

$\{$use $(A\otimes B)(C\otimes D) = AC\otimes BD\}$

$$(5) \quad \boxed{0 = \bar{X}'(I\otimes Z'(Z'Z)^{-1})(\Sigma^{-1}\otimes Z')(\bar{X}\,\hat{\bar{\delta}}_{3SLS} - \bar{y})}$$

Compare this to:

$$(4) \quad \boxed{0 = \bar{D}(\Sigma^{-1}\otimes Z')(\bar{X}\,\hat{\bar{\delta}}_{FIML} - \bar{y})}$$

3SLS and FIML both use a matrix of weights to form instrumental variables

$$\hat{\bar{\delta}} = (\bar{W}'\bar{X})^{-1}(\bar{W}'\bar{y})$$

$$\bar{W}_{FIML} = (I\otimes Z)\underbrace{(\Sigma^{-1}\otimes I)\,\bar{D}}_{\hat{A}_{FIML}}$$

$$\bar{W}_{3SLS} = (I\otimes Z)\underbrace{(\Sigma^{-1}\otimes I)(I\otimes(Z'Z)^{-1}Z')\,\bar{X}}_{\hat{A}_{3SLS}}$$

Look at the plim of $(I \otimes (Z'Z)^{-1}Z')\bar{X}$:

$$\operatorname{plim} (I \otimes (Z'Z)^{-1}Z')\bar{X} = \operatorname{plim} \begin{bmatrix} \left(\frac{Z'Z}{T}\right)^{-1}\left(\frac{Z'X^1}{T}\right) & & 0 \\ & \ddots & \\ 0 & & \left(\frac{Z'Z}{T}\right)^{-1}\left(\frac{Z'X^m}{T}\right) \end{bmatrix}$$

Now, $\operatorname{plim} \frac{Z'X^j}{T} = \operatorname{plim} \frac{Z'[Y^j, Z^j]}{T}$

$$= \operatorname{plim} \frac{Z'[Z\Pi_j + V_j \ , \ Z C_j]}{T}$$

$$= \operatorname{plim} \left(\frac{Z'Z}{T}\right) [\Pi_j \ , \ C_j]$$

$$= \operatorname{plim} \left(\frac{Z'Z}{T}\right) D_j$$

$$\Rightarrow \operatorname{plim} (I \otimes (Z'Z)^{-1}Z')\bar{X} = \begin{bmatrix} D_1 & & \\ & \ddots & \\ & & D_m \end{bmatrix} = \bar{D}$$

This shows that the weighting matrices

$\bar{W}_{FIML}$, $\bar{W}_{3SLS}$ are asymptotically the same.

$\Rightarrow$ FIML, 3SLS are asymptotically equivalent.

Limited Information Maximum Likelihood (LIML)

LIML is simply FIML on only one equation. We do FIML on:

This is our system

$$\begin{cases} \underset{T\times 1}{y_1} = \underset{T\times r_1}{Y_1}\,\underset{r_1\times 1}{\beta_1} + \underset{T\times s_1}{Z_1}\,\underset{s_1\times 1}{\gamma_1} + \underset{T\times 1}{U_1} & \text{(1st equation)} \\ \underset{T\times r_1}{Y_1} = \underset{T\times K}{Z}\,\underset{K\times r_1}{\Pi_1} + \underset{T\times r_1}{V_1} & \left(\begin{array}{l}\text{Reduced Form for endog.} \\ \text{Variables on RHS of 1st eq.}\end{array}\right) \end{cases}$$

For a single observation,

$$Var\begin{pmatrix} u_{1t} \\ v_{1t} \end{pmatrix} = \Psi = \underset{(1+r_1)\times(1+r_1)}{\begin{pmatrix} \underset{1\times 1}{\sigma_{11}} & \underset{1\times r_1}{\Psi_{12}} \\ \underset{r_1\times 1}{\Psi_{21}} & \underset{r_1\times r_1}{\Psi_{22}} \end{pmatrix}}$$

$$Var\begin{pmatrix} U_1 \\ V_1 \end{pmatrix} = \Psi \otimes I_T$$

$$\Psi^{-1} = \begin{pmatrix} \Psi^{11} & \Psi^{12} \\ \Psi^{21} & \Psi^{22} \end{pmatrix}$$

where $\Psi\Psi^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}$.

Lets use the FOC's found for FIML:

$$\Pi_j' Z' U \Sigma_{(j)}^{-1'} = 0 \quad \text{(for endog. RHS variables)}$$

$$Z_j' U \Sigma_{(j)}^{-1'} = 0 \quad \text{(for exog. RHS variables).}$$

Applying these to the first equation and noting that $U = [u, V_1]$, $\Sigma_{(1)}^{-1} = \begin{bmatrix} \psi^{11} \\ \psi^{21} \end{bmatrix}$, we have

$$\Pi_1' Z' U \Sigma_{(1)}^{-1'} = 0$$

$$\Pi_1' Z' [u, V_1] \begin{pmatrix} \psi^{11} \\ \psi^{21} \end{pmatrix} = 0$$

$$\Rightarrow \boxed{\text{FOC for } \beta_1: \quad \Pi_1' Z' [u, \psi^{11} + V_1 \psi^{21}] = 0} \quad (L1)$$

$$Z_1' U \Sigma_{(1)}^{-1'} = 0$$

$$\Rightarrow \boxed{\text{FOC for } \gamma_1: \quad Z_1' (u, \psi^{11} + V_1 \psi^{21}) = 0} \quad (L2)$$

The second block of equations in the system is $Y_1 = Z\Pi_1 + V_1$.

Here, we have only exogenous variables on the RHS.

$$Z' U \Sigma_{(2)}^{-1'} = 0$$

$$\Rightarrow \boxed{\text{FOC for } \Pi_1: \quad Z' (u, \psi^{12} + V_1 \psi^{22}) = 0} \quad (L3)$$

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From (L3), we have:

$$Z'U_1 \psi^{12} + Z'V_1 \psi^{22} = 0$$

$$\Rightarrow \quad Z'V_1 = -Z'U_1 \psi^{12} (\psi^{22})^{-1}$$

Plug this into (L1):

$$\pi_1' Z' U_1 \psi^{11} + \pi_1' (-Z'U_1 \psi^{12} (\psi^{22})^{-1}) \psi^{21} = 0$$

$$\Rightarrow \quad \pi_1' Z' U_1 (\underbrace{\psi^{11} - \psi^{12} (\psi^{22})^{-1} \psi^{21}}_{\text{Scalar}}) = 0$$

$$\Rightarrow \quad \boxed{\pi_1' Z' U_1 = 0} \quad (L1')$$

Also, plug into (L2) $Z_1' V_1 = -Z_1' U_1 \psi^{12} (\psi^{22})^{-1}$

(this is also true from L3):

$$Z_1' U_1 \psi^{11} - Z_1' U_1 \psi^{12} (\psi^{22})^{-1} \psi^{21} = 0$$

$$\Rightarrow \quad Z_1' U_1 (\underbrace{\psi^{11} - \psi^{12} (\psi^{22})^{-1} \psi^{21}}_{\text{Scalar}}) = 0$$

$$\Rightarrow \quad Z_1' U_1 = 0$$

Define the selection matrix C_1 by $Z_1 = Z C_1$

Then we have:

$$\boxed{C_1' Z' U_1 = 0} \quad (L2')$$

Stack (L1'), (L2'):

$$\pi_1' Z' U_1 = 0$$
$$C_1' Z' U_1 = 0 \qquad \text{Define } D_1 = [\pi_1 \;\; C_1]$$

$$\Rightarrow \boxed{D_1' Z' U_1 = 0}$$ This is the FOC for LIML.

Plugging in $U_1 = y_1 - X_1 \delta_1$,

we have $\boxed{\hat{\delta}_{1,LIML} = (D_1' Z' X)^{-1} (D_1' Z' y)}$

Note that $D_1 = [\pi_1 \;\; C_1]$

π_1 must be estimated and plugged in.

- LIML is asymptotically equivalent to 2SLS (showing this is similar to FIML, 3SLS).

TESTING

Limited Information

I) Test of Overidentifying Restrictions

The general test where u may be nonspherical, $Z'u/\sqrt{T} \xrightarrow{d} N(0, V)$,

$$V = \text{plim}\ \frac{Z'UU'Z}{T} = \text{plim}\ \frac{Z'\Omega Z}{T}$$

$$\Omega = E(uu')$$

is given by:

$$W = \left(\frac{Z'\hat{u}}{\sqrt{T}}\right)' \hat{V}^{-1} \left(\frac{Z'\hat{u}}{\sqrt{T}}\right) \xrightarrow{d} \chi^2_{l-k}$$

where $\hat{V} \xrightarrow{p} V$ and $l-k$ = degree of overidentification ($y_j = \underset{k}{X_j}\, \delta_j + u_j$, $\underset{l}{Z}$)

When u is spherical,

$$V = \sigma^2\ \text{plim}\ (Z'Z/T) \quad \text{and}$$

$$\hat{V} = \hat{\sigma}^2 (Z'Z/T) = (\hat{u}'\hat{u}/T)(Z'Z/T)$$

Plugging in,

$$W = T\ \frac{\hat{u}'Z(Z'Z)^{-1}Z'\hat{u}}{\hat{u}'\hat{u}} \xrightarrow{d} \chi^2_{l-k}$$

TESTING - LI

I) Note that $\dfrac{\hat{u}'Z(Z'Z)^{-1}Z'\hat{u}}{\hat{u}'\hat{u}}$ is the R^2 in the regression of $\hat{u}$ on Z.

So an equivalent test is:

1) Regress $\hat{u}$ on Z, obtain R^2
2) TR^2 is our test statistic, with distribution $\chi^2_{\ell-k}$

II) Hausman Test : OIV vs. 2SLS

$$\hat{q} = \hat{\delta}_{2SLS} - \hat{\delta}_{OIV}$$

$$h = T\hat{q}'[V(\hat{\delta}_{2SLS}) - V(\hat{\delta}_{OIV})]\hat{q} \xrightarrow{d} \chi^2_k$$

where k is the dimension of δ

III) Tests for Identification based on RF

Structural Form overidentifying restrictions impose non linear restrictions on the RF parameters : $h(\pi) = 0$

We test $H_0: h(\pi) = 0$ on the RF using Wald, LM, LR Tests.

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TESTING

FULL INFORMATION

$\bar{y} = \bar{X}\bar{\delta} + \bar{u}$ stacked model

$p_i = l - k_i$ is degree of overidentification of the i^{th} equation.

$$W = \hat{u}'(\hat{\Sigma}^{-1} \otimes P_Z)\hat{u} \xrightarrow{d} \chi^2_{\underbrace{\sum_{i=1}^{m} p_i}}$$

degree of overidentification of the system

This can be written as

$$W = \frac{\hat{u}'(I \otimes Z)}{\sqrt{T}} \underbrace{\left[\hat{\Sigma} \otimes \left(\frac{Z'Z}{T}\right)\right]^{-1}}_{} \frac{(I \otimes Z)'\hat{u}}{\sqrt{T}}$$

inverse of Avar of $\dfrac{(I \otimes Z)'\bar{u}}{\sqrt{T}}$

which helps to explain the distribution of W.

If we do FIML, we can use a LR test based on the RF:

$$W = 2\left[\log L(\hat{\pi}_u, \hat{\Omega}_u) - \log L(\hat{\pi}_R, \hat{\Omega}_R)\right] \xrightarrow{d} \chi^2_{(p)}$$

overidentification of the system

We can plug in

$$\hat{\pi}_R = -\hat{\Gamma}_{FIML}\hat{\beta}^{-1}_{FIML}$$

$$\hat{\Omega}_R = \hat{\beta}^{-1\prime}_{FIML}\hat{\Sigma}_{FIML}\hat{\beta}^{-1}_{FIML}.$$

Computing $\hat{\pi}_R$ in this way, we can also do a Wald test as in LI.