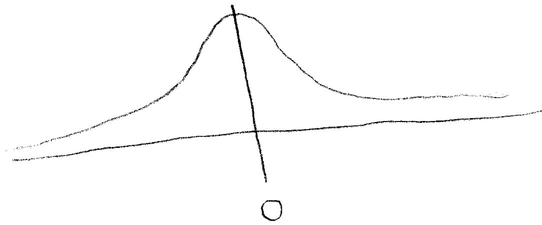
$t = \frac{\overline{X} - \mu}{s}$ "Never know exactly that a rv is normal, but you don't know scale."

$$f_T(t) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \frac{1}{(p\pi)^{1/2}} \frac{1}{(1+t^2/p)^{\frac{p+2}{2}}}$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$

T is symmetric and bell shaped

- $p-1$ moments exist
- $p=1$ Cauchy
- $p \to \infty \Rightarrow T \to$ normal



Asymptotic Theory

$T(\overline{X})$ is a function of rvs. What is the distribution of $T(\overline{X})$?

Truth: $X_1, \ldots, X_n$ - finite amount of data

approximation: $X_1, X_2, \ldots$ - infinite data

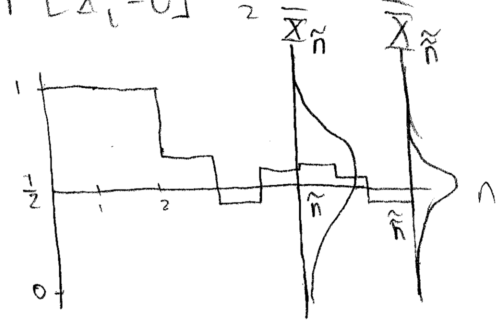
Laws of Large Numbers

$\overline{X}_n$ (random variable) μ (scalar) What is the relationship of these?

Usual ideas of convergence:

$$\sum_{i=0}^{n} \beta^i \to \frac{1}{1-\beta} \quad \text{for } |\beta| < 1$$

Suppose $X_i = \begin{cases} 1 & Pr[X_i = 1] = \frac{1}{2} \\ 0 & Pr[X_i = 0] = \frac{1}{2} \end{cases}$



Let $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$

$(\bar{X}_1, \bar{X}_2, \ldots)$ takes on a path $(\bar{x}_1, \bar{x}_2, \ldots)$

Thm 3.6.1 (Chebyshev's inequality). Let X be a r.v. Let $g(X)$ be a nonnegative function. Then

$$Pr[g(X) \geq r] \leq \frac{E[g(X)]}{r}$$

Pf:

$$E[g(X)] = \int_{\mathbb{R}} g(x) f(x) dx = \int_A g(x) f(x) dx + \int_{A^c} g(x) f(x) dx$$

where $A = \{x : g(x) \geq r\}$

$$\Rightarrow E[g(X)] \geq r \int_A f(x) dx = r \, Pr[g(X) \geq r]$$

$$\Rightarrow Pr[g(X) \geq r] \leq \frac{E[g(X)]}{r} \quad \square$$

Remarks:
- Can now go between moments and probabilities
- $X \sim D(\mu, \sigma^2)$, where D is an unspecified distribution.

Let $g(x) = \left[\frac{x - \mu}{\sigma}\right]^2$

applying Chebyshev:

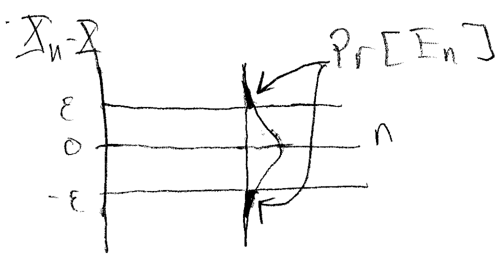
$$\Pr\left[\left(\frac{X-\mu}{\sigma}\right)^2 \geq t^2\right] \leq \frac{E\left[\left(\frac{X-\mu}{\sigma}\right)^2\right]}{t^2} = \frac{1}{t^2}$$

$$\Rightarrow \Pr[|X-\mu| \geq t\sigma] \leq \frac{1}{t^2}$$

- Let $t=2$. Then $\Pr[|X-\mu| \geq 2\sigma] \leq \frac{1}{4}$.
  - ie prob. of being more than 2 standard deviations away from the mean is less than a quarter.
  - Note that this is predicated on the existence of the variance.
- special case: $g(X) = |X-\mu|$. This is known as Markov's inequality.

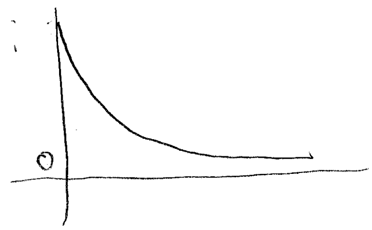
<u>Defn 5.5.1</u>: a sequence of random variables <u>converges in probability</u> to a rv X if $\forall \varepsilon > 0$,

$$\lim_{n\to\infty} \Pr[|X_n - X| \geq \varepsilon] = 0.$$

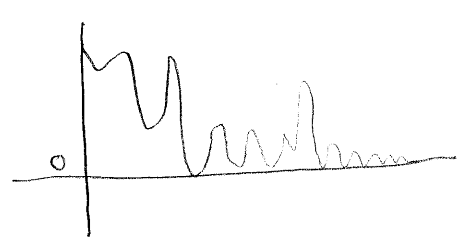


- Let $E_n = |X_n - X| > \varepsilon$.

Could have:



or



- Convergence in probability is a weak notion.

Thm 5.5.2 (WLLN): Let $X_1, X_2, \ldots$ be iid rvs with $E[X_1] = \mu$, $Var(X_1) = \sigma^2$. Define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

Then $\lim_{n\to\infty} \Pr[|\bar{X}_n - \mu| > \varepsilon] = 0$. Thus $\bar{X}_n \xrightarrow{P} \mu$.

Proof: For every $\varepsilon > 0$, $\Pr[|\bar{X}_n - \mu| > \varepsilon] = \Pr[(\bar{X}_n - \mu)^2 > \varepsilon^2]$

$$\leq \frac{E[(\bar{X}_n - \mu)^2]}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \longrightarrow 0 \text{ as } n \to \infty.$$

Thus, $\bar{X}_n \xrightarrow{P} \mu$. $\square$

e.g. coin flipping $X_i = \begin{cases} 1 & \Pr[X_i = 1] = \frac{1}{2} \\ 0 & \Pr[X_i = 0] = \frac{1}{2} \end{cases}$

$E[X_i] = \frac{1}{2}$ $\quad Var(X_i) = \frac{1}{4} < +\infty$. Thus, $\bar{X}_n \xrightarrow{P} \frac{1}{2}$

Remarks: 1] Yet again, averaging removes errors

2] No distributional assumption. Just need variance existence.

- There is a trade-off between existence of variance and independence.

"No one has seen an unbounded random variable" - Hal White.

Defn (Not in book): a sequence of random variables $X_1, X_2, \ldots$ converges in quadratic mean to X if $\lim_{n\to\infty} E[(X_n - X)^2] = 0$

We write $X_n \xrightarrow{qm} X$.

Claim: $X_n \xrightarrow{q.m} X \Rightarrow X_n \xrightarrow{P} X$

- This is just one example of convergence in r^{th} mean.
- $X_n \xrightarrow{P} X \not\Rightarrow X_n \xrightarrow{qm} X$, since variances may not exist.
- Eg $X_n = \begin{cases} 0 & Pr[X_n = 0] = 1 - \frac{1}{n} \\ n & Pr[X_n = n] = \frac{1}{n} \end{cases}$

$E[X_n] = 1$, $E[X_n^2] = 0^2(1 - \frac{1}{n}) + n^2 \cdot \frac{1}{n} = n \to \infty$

$\Rightarrow X_n$ does not converge in quadratic mean

Defn: a statement about random variables holds almost surely (a.s.) if $\exists$ an event N where $Pr[N] = 0$ such that statement holds $\forall x \in N^c$.

Remarks: [1] Can ignore extremely rare events
[2] almost everywhere (a.e.)

Defn: 5.5.6: a sequence of rvs $X_1, X_2, \ldots$ converges almost surely to a rv X if $\forall \varepsilon > 0$,
$Pr[\lim_{n \to \infty} |X_n - X| > \varepsilon] = 0$. Write $X_n \xrightarrow{as} X$.

$E_n = \{|X_n - X| > \varepsilon\}$. $\lim_{n \to \infty} E_n$

In general,

- $A_1 \subset A_2 \subset A_3 \subset \ldots$ (nondecreasing)

$\Rightarrow \lim A_i = \bigcup_{i=1}^{\infty} A_i$

b) $A_1 \supset A_2 \supset \dots$ (nonincreasing)

$\Rightarrow \lim_{n \to \infty} A_i = \bigcap_{i=1}^{\infty} A_i$

c) $\sup_{i \geq I} A_i = \bigcup_{i=I}^{\infty} A_i$

d) $\inf_{i \geq I} A_i = \bigcap_{i=I}^{\infty} A_i$

eg. ∞ sequence of iid coin tosses

outcomes: $\{HHTTT\,HHT \dots\}$

- Let A_i = Head on i^{th} toss

$\Rightarrow A_1 = \{H, x_2, x_3, \dots\}$ $x_j \in \{H,T\}$

$A_2 = \{x_1, H, x_3, \dots\}$ $x_j \in \{H,T\}$

- $\sup_{i \geq 1} A_i = \bigcup_{i=1}^{\infty} A_i = \{\text{at least one head}\}$

$= \{T, T, T, \dots\}^c$

- $\inf_{i \geq 1} A_i = \bigcap_{i=1}^{\infty} A_i = \{H, H, \dots\}$

- $\sup_{i \geq I} A_i = \bigcup_{i=I}^{\infty} A_i = \{\text{at least one head after } I\}$

$\sup_{i \geq I} A_i \supset \sup_{i \geq I+1} A_i \supset \dots$ (non increasing)

- $\inf_{i \geq I} A_i = \bigcap_{i=I}^{\infty} A_i = \{x_1, \dots, x_{i-1}, H, H, \dots\}$

$\inf_{i \geq I} A_i \subset \inf_{i \geq I+1} A_i \subset \dots$ (nondecreasing)

$$\limsup A_i = \bigcap_{I=1}^{\infty} \bigcup_{i=I}^{\infty} A_i$$

$$\liminf A_i = \bigcup_{I=1}^{\infty} \bigcap_{i=I}^{\infty} A_i$$

We say that $\lim A_i$ is $\liminf A_i = \limsup A_i$ when those are equal.

We are interested in $\limsup E_n$.