This course will go into Bayesian statistics

Review of probability

Random variable $X : \mathcal{X} \to [0,1]$

$(\underbrace{\mathcal{X}}_{\text{outcome space}}, \underbrace{\mathcal{F}}_{\sigma\text{-algebra on } \mathcal{X}}, \underbrace{P}_{\text{prob. measure}})$ is a probability space

Represent info. about X in two ways. $(X(\omega) = x,\ \omega \in \mathcal{X})$

$x \in X(\mathcal{X})$ is an atom

a) pdf

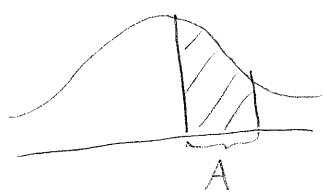
b) cdf

pdf: $f_X(x)$ gives probabilities (P) to outcomes x

cdf: $F_X(x) = \Pr[X \le x] \equiv P(X^{-1}((-\infty, x]))$

X is continuous if $F_X(x)$ is continuous

$\Pr[X \in A] = \int_A f_X(x)\,dx$



Support of X: $\text{supp}(X) = \{x : f_X(x) > 0\}$

$Y = g(X)$ can make new rvs out of old ones

└ g needs to be a measurable function

Expectation

$$E[g(X)] = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x)\,dx & \text{if } X \text{ is continuous} \\ \sum_x g(x) f_X(x) & \text{if } X \text{ is discrete} \end{cases}$$

Moments need not exist
If $E[|g(X)|] < +\infty$, then the moment exists

Expectation properties are just integral properties.
- $E[ag(X)+b] = aE[g(X)]+b$
- $E[ag_1(X)+bg_2(X)] = aE[g_1(X)]+bE[g_2(X)]$
- $E[g(X)] \neq g(E[X])$ Jensen's inequality demonstrates this in special cases

Moments

Mean: $\mu = E[X]$
n^{th} moment: $\mu'_n = E[X^n]$
central moment $\mu_n = E[(X-\mu)^n]$
eg variance: $\sigma_X^2 = \mu_2 = E[(X-\mu)^2]$

If $E[|X|^n] < +\infty$, $E[|X|^{n-j}] < +\infty$ $j = 1, \dots, n-1$
- The more moments that exist, the faster the pdf "tails down."

For all bounded distributions, all moments exist

MGFs

Spse X is an rv

$M_X(t) = E[e^{tX}]$
Need $E[e^{tX}] < +\infty$ for t near 0 for $M_X(t)$ to exist.

$M_X(t)$ only exists if X has an infinite number of moments.

eg. $X \sim$ exponential(β) $\qquad \beta > 0$

$f_X(x) = \frac{1}{\beta} e^{-x/\beta} \mathbb{1}_{\{0 \le x < +\infty\}}$

$$M_X(t) = \int_0^\infty e^{tx} \frac{1}{\beta} e^{-x/\beta} dx$$

$$= \frac{1}{\beta} \int_0^\infty e^{(tx - x/\beta)} dx$$

$$= \frac{1}{\beta} \int_0^\infty e^{-x\left(\frac{1}{\beta} - t\right)} dx \qquad \frac{1}{\beta} - t = \frac{1-\beta t}{\beta}$$

$$= \frac{1}{\beta} \int_0^\infty e^{-x\left(\frac{\beta}{1-\beta t}\right)} dx$$

$$= \frac{1}{\beta} \frac{\beta}{1-\beta t} \underbrace{\int_0^\infty \frac{1-\beta t}{\beta} e^{-x\left(\frac{\beta}{1-\beta t}\right)} dx}_{=1}$$

$$= \frac{1}{1-\beta t}$$

$$E[X^n] = \frac{d^n}{dt^n}\bigg|_{t=0} M_X(t)$$

Characteristic functions

$\varphi_X(t) = E[e^{itX}]$

- always exists
- characteristic functions are unique

Transforms/functions of random variables

$Y = g(X)$

(1) calculus tools (transforms)

- g must be monotone (need the inverse to exist)
- X must be continuous
- $g \in C^1(\text{supp}(X))$

$$f_Y(y) = \begin{cases} f(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right| & y \in \text{supp}(Y) \\ 0 & \text{else} \end{cases}$$

(2) MGFs

- $M_Y(t) = E[e^{tg(X)}]$

Multivariate Extensions

X, Y $\quad A \subset \mathbb{R}^2$

$$Pr[(X,Y) \in A] = \int_A \underbrace{f_{X,Y}(x,y)}_{\text{joint pdf}}\, dx\, dy$$

Marginal pdf. $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$ "integrate out Y"

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$

Conditionals: $f_Y(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$ $\quad f_X(x) > 0$

Independence: $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$ $\quad \forall x, y$

Conditional Expectations

$$E[Y \mid X = x] = \int_{\mathbb{R}} y \, f_{Y|X}(y|x) \, dy = h(x)$$

$$E\left[E[Y|X]\right] = \int_{\mathbb{R}} \int_{\mathbb{R}} y \, f_{Y|X}(y|x) \, dy \, f_X(x) \, dx$$

$$= \int_{\mathbb{R}^2} y \, f_{X,Y}(x,y) \, dx \, dy$$

$$= E[Y]$$

Law of Iterated Expectations