3rd problem set due November 8th.

Dynamic principle agent problems

Holmström-Milgrom (EMA 1987): "Aggregation and Linearity in the Provision of Intertemporal Incentives."

- Mirrlees gives us that highly nonlinear contracts can be optimal. (Mirrlees non-existence result.)
  - This is troublesome, since it is very sensitive to certain assumptions about preferences or distributions
- Linear schemes seem more robust and natural. This paper provides justification.

- T periods
- x_1 outcome in period 1: $x_1 \in \{0,1\}$ sale/no sale

  ⋮

  x_T outcome in period T

assumptions

1. agent chooses $p^t(h_t) = Pr[x_t = 1]$ at period t.
   - can be history dependent.
2. all periods are technologically identical. (ie error term, cost of effort stay same.)
3. Cost of "effort" is per period and is given by $c(\cdot)$ which is increasing and convex
4. Principal commits to a sharing scheme $s(x_1, \ldots, x_T)$
   - get paid at time T

5] agents' utility is $u(s) = -e^{-rs}$
- CARA $\Rightarrow$ no wealth effects

We will assume $T=2$.

Timing

Period 1: agent chooses $p^1(\overbrace{h_1}^{\text{null history}})$ at cost $c(p^1)$

Period 2: agent chooses $p^2(h_2) = p_i^2$ where $i = \begin{cases} 1 & \text{if } x_1 = 1 \\ 0 & \text{if } x_1 = 0 \end{cases}$

Defn: "constant pressure" if agent is rewarded for success no matter when it occurs.

Define: $\omega_i(p_i^2, s_{i0}, s_{i1})$ $i = 0, 1$ solves

$$u(\omega_i) = p_i^2 u(s_{i1} - c(p_i^2)) + (1-p_i^2) u(s_{i0} - c(p_i^2))$$

- This is the CE in the branch associated with outcome i in pd 1.

IC constraints:

$$(IC_1)\quad p^1 = \arg\max_{\hat{p}^1} \{ \hat{p}^1 u(w_1 - c(\hat{p}^1)) + (1-\hat{p}^1) u(w_0 - c(\hat{p}^1)) \}$$

$$(IC_{2i})\quad p_i^2 = \arg\max_{\hat{p}_i^2} \{ \hat{p}_i^2 u(s_{i1} - c(p^1) - c(\hat{p}_i^2)) + (1-\hat{p}_i^2) u(s_{i0} - c(p^1) - c(\hat{p}_i^2)) \}$$

Two constraints

$$(IR\ \text{exante})\quad p^1 p_1^2 u(s_{11} - c(p^1) - c(p_1^2)) + p^1(1-p_1^2) u(s_{10} - c(p^1) - c(p_1^2))$$
$$+ (1-p^1) p_0^2 u(s_{01} - c(p^1) - c(p_0^2)) + (1-p^1)(1-p_0^2) u(s_{00} - c(p^1) - c(p_0^2))$$
$$\geq u(0)$$

Principal's problem

$$\max\ p_1 p_1^2(x_1+x_1-s_{11}) + p_1(1-p_1^2)(x_1+x_0-s_{10})$$
$$+ (1-p_1)p_0^2(x_0+x_1-s_{01}) + (1-p_1)(1-p_0^2)(x_0+x_0-s_{00})$$
$$\text{st}\quad IR,\ IC_1,\ IC_{21},\ IC_{22},\ \omega_i(p_i^2, s_{i0}, s_{i1}) = w_i$$

To solve this,

(a) $w_1 - w_0$ determines p'

(b) IR binds and this determines intercept of sharing rule. (Need CARA here)

(c) Take desired CE levels $w_1 = \hat{w}_1$, $w_0 = \hat{w}_0$ as exogenous

Then find best incentives p_i^2's in period 2.

The problem becomes (auxilliary program) (AUX)

$$\max_{p_i^2, s_{i1}, s_{i0}}\ p_i^2(x_i+x_1-s_{i1}) + (1-p_i^2)(x_i+x_0-s_{i0})$$
$$\text{s.t.}\quad IC_{2i},\ \omega_{i0}(p_i^2, s_{i0}, s_{i1}) = \hat{w}_i$$

<u>Lemma 1:</u> If (p_i^2, s_{i1}, s_{i0}) solves (AUX), then p_i^2 and $s_{i1}-s_{i0}$ are independent of $\hat{w}_i$.

<u>Implication:</u> The agent gets up one day and faces his problem. The sol'n to this does not depend on the level of CE wealth

This implies $s_{11} - s_{10} = s_{01} - s_{00}$ in principal's problem no matter what w_1, w_0 are.

Lemma 2: The principal's first period problem is technologically the same as each second period branch.

Sketch of proof: Let the expected joint surplus after a period 1 success in CE units be $\pi(1)$ and following a period 1 failure be $\pi(0)$. The only difference for the principal is the desired CE to give to the agent. By lemma 1, joint surplus does not depend on his wealth. It just determines the split. Therefore,

$$\pi(1) - \pi(0) = w_1 - w_0.$$

Therefore, principal's period 1 problem is to

$$\max_{p^1, w_1, w_0} p_i(x_1 + \pi(0) + w_1 - w_0) + (1-p_i)(x_0 + \pi(0))$$

s.t. IC1

This has same soln as AUX up to a choice of w.

(a) IR pins down w_0

(b) For period 2, sharing rules to provide the incentive $w_1 - w_0$ in period 1, need

$$w_1 - w_0 = \omega_1 (p_1^2, s_{11}, s_{10}) - \omega_0 (p_0^2, s_{01}, s_{00}).$$

(c) The same p is optimal in period 1. That is, $p_1^2 = p_0^2$. The best $w_1 - w_0$ satisfies

$$w_1 - w_0 = s_{11} - s_{10} = s_{01} - s_{00}$$

Let a be the reward for success. Then

$a = s_{11} - s_{10} = s_{01} - s_{00} = w_1 - w_0$, and $\underline{w_1 - w_0 = s_{11} - s_{01}}$ (by stationarity)

$s_{11} = s_{10} + s_{01} - s_{00} = 2s_{01} - s_{00}$

$\Rightarrow s_{10} = s_{01} = s_{00} + a$

and $s_{11} = s_{10} + a = s_{00} + 2a$

Let $z = \#$ successes, $x^T = (x_1, \ldots, x_T)$ be the history of success. Then $s(x^T) = s(z(x^T)) = az + \underbrace{b}_{\text{ct } s_{00}}$

Next, let us go to continuous time:

$dz(t) = \mu dt + \sigma dB(t)$. Take $T \to \infty$, and so optimal scheme is linear.

Remarks:

1. This is incredible.
2. CARA is important.
   - If not CARA, coefficient of risk aversion changes, so the incentive scheme must change. (wrt successes and failures.)
3. Continuous time interpretation. (Hellwig - Schmidt EMA'02)
   Martin
4. No intermediate consumption
   - kills off precautionary savings motives
5. This is a justification.

Sannikov - continuous time principal-agent model
- working paper