Exam: December 12th, 4:00-6:00pm

Monotone Comparative Statics

Recall: From first order approach, we got:

$$\frac{1}{u'(\cdot)} = \mu + \lambda \frac{f_a(\cdot)}{f(\cdot)}$$

- There is nothing here to suggest linear incentive schemes.

- Under what conditions do we get nice, linear incentive schemes?
  - This is where monotone comparative statics fits in.

- Risk neutral principal
- Risk averse agent
- φ - some parameter
- $q_1(\varphi), \ldots, q_n(\varphi)$
- $\pi_1(a), \ldots, \pi_n(a)$
- $a \in A = \mathbb{R}^n$
- agents' utility: $\underbrace{u(a, I)}_{\text{payments from principal}} = G(a) + K(a)v(I)$

(A1): v is a continuous, strictly increasing, real valued, concave function on an open ray $(\underline{I}, +\infty)$. Let $\lim_{I \to \underline{I}} v(I) = -\infty$

assume that G and K are continuous, real-valued functions and $K(a) > 0$ $\forall a$.

For all $a_1, a_2 \in A$, $I, \hat{I} \in \mathcal{I} \equiv (\underline{I}, +\infty)$

$$G(a_1)+K(a_1)v(I) \geq G(a_2)+K(a_2)v(I)$$

$$\Rightarrow G(a_1)+K(a_1)v(\hat{I}) \geq G(a_2)+K(a_2)v(\hat{I})$$

- Preferences for income lotteries are independent of action.

(A2) $\pi_i(a) > 0 \quad \forall a \in A$ and $i=1,\dots,n$

- rules out the moving support situation

an <u>incentive scheme</u> is an n-dimensional vector $I_1, \dots, I_n$ where agent gets I_i if q_i occurs.

Grossman-Hart: $c(a)$ - lowest cost of implementing action a.

<u>Step 2:</u> $\max_{a \in A} \sum_{i=1}^{n} \pi_i(a) q_i(\varphi) - c(a)$

Let $a^{**}(\varphi; c) = \arg\max_a \{B(a,\varphi) - c(a)\}$

<u>Proposition:</u> $a^{**}(\varphi; c)$ is nondecreasing (in strong set order) in φ for all functions c iff B has increasing differences.

$B(a,\varphi) = \sum_{i=1}^{n} \pi_i(a) q_i(\varphi)$ has increasing differences if $\sum_{i=1}^{n} \pi_i'(a) q_i'(\varphi)$, where $a_1 > a_2 \Leftrightarrow c^{FB}(a_1) > c^{FB}(a_2)$

Intuition behind this result: Suppose two outcomes: $\{H, L\}$.

(1) $\pi_L'(a) q_L'(\varphi) + \pi_H'(a) q_H'(\varphi) \geq 0$.

- $\pi_L(a) + \pi_H(a) = 1$

$\Rightarrow \pi_L'(a) + \pi_H'(a) = 0$

$\Rightarrow \pi_H'(a) = -\pi_L'(a)$

$\Rightarrow$ (1) becomes: $\pi_L'(a) q_L'(\varphi) - \pi_L'(a) q_H'(\varphi) \geq 0$

$$\Leftrightarrow \underbrace{\pi_L'(a)}_{(-)} [q_L'(\varphi) - q_H'(\varphi)] \geq 0$$

by First order stochastic dominance ("CDF is weakly lower")

$$\Leftrightarrow q_L'(\varphi) - q_H'(\varphi) < 0$$

$$\Leftrightarrow q_L'(\varphi) < q_H'(\varphi)$$

- want to induce higher action when the higher action is more desirable.
- can have benefit function look like: $B = \sum_{i=1}^{n} \pi_i(a, \theta) q_i(\varphi)$

Value of information

Holmstrom (Bell, 1979)

q - output
a - action
s - signal

When do we want to condition the optimal incentive scheme on s?

- States: $\pi_{is}(a)$ = prob state i occurs given we saw signal s, conditional on action

$$= \pi(i, s|a)$$

$$\frac{1}{u'(w)} = \lambda + \mu \underbrace{\frac{\pi'_{is}(a)}{\pi_{is}(a)}}_{\equiv \frac{g_a}{g}}$$

- When is $\dfrac{g_a(q,s|a)}{g(q,s|a)} = \dfrac{f_a(q|a)}{f(q|a)}$?

- There exist functions $m(q|a)$ and $n(q|s)$ such that $g(q,s|a) = m(q|a)\,n(q|s)$ iff

  q is a sufficient statistic for the pair (q,s) with respect to a.

  - all the information for (q,s) is embodied in q.
  - Halmos-Savage Factorization criterion.

Random Schemes

Suppose instead of getting $\underbrace{I_1, \ldots, I_n}^{\text{deterministic}}$, you get a lottery on $I_1, \ldots, I_n$. (ie: get $\underset{p=\frac{1}{2}}{\frac{1}{2}I_j}$, $\underset{p=\frac{1}{2}}{2I_j}$ instead of I_j)

- Will not make us better off.

Linear Contracts: (Restriction of contract space)

- $w = t + vq$
- $q = a + \varepsilon$ $\quad a \in A = \mathbb{R}$, $\varepsilon \sim N(0, \sigma^2)$
- Principal risk neutral
- agent risk averse with exponential utility fcn: CARA preferences:

$$u(w, q) = -e^{-r(w - \psi(a))} \qquad \psi(a) = \tfrac{1}{2} c a^2$$

- Principal chooses a, t, v

$$\max_{a,t,v} E[q - w]$$

s.t. i) $a \in \arg\max_a E\left[-e^{-r(w - \frac{1}{2}(a^2))}\right]$

ii) $E\left[-e^{-r(w - \frac{1}{2}(a^2))}\right] \geq -e^{-r\bar{w}}$

Recall: $E[e^{\gamma X}] = \exp\left\{\frac{\gamma^2 \sigma^2}{2}\right\}$ if $X \sim N(0, \sigma^2)$

$$\Rightarrow E\left[-e^{-r(w - \psi(a))}\right] = E\left[-e^{-r(t + va + v\varepsilon - \psi(a))}\right]$$

$$= -\exp\{-r(t + va - \psi(a))\} E[e^{-rv\varepsilon}]$$

$$= -e^{-r\hat{w}(a)}$$

where $\hat{w}(a) = t + va - \frac{r}{2} v^2 \sigma^2 - \frac{1}{2} c a^2$

$\Rightarrow$ (i) becomes: $v - ca = 0 \Rightarrow a = \frac{v}{c}$

Problem becomes:

$$\max_{v,t} \left\{ \frac{v}{c} - t - \frac{v^2}{c} \right\}$$

s.t. $\hat{w}(a) = \hat{w}\left(\frac{v}{c}\right) = \bar{w}$

(IR): $t + \frac{v^2}{c} - \frac{r}{2}v^2\sigma^2 - \frac{v^2}{2c} = 0$

$$\Rightarrow \quad t + \frac{v^2}{2c} - \frac{r}{2}v^2\sigma^2 = \bar{w}$$

$$\Rightarrow \max_{v} \left\{ \frac{v}{c} - \frac{v^2}{c} + \frac{v^2}{2c} - \frac{r}{2}v^2\sigma^2 - \bar{w} \right\}$$

FOC: $\frac{1}{c} - \frac{2v}{c} + \frac{v}{c} - rv\sigma^2 = 0$

$$\Rightarrow \quad v\left(\frac{1}{c} - \frac{2}{c} - r\sigma^2\right) = \frac{-1}{c}$$

$$v(-1 - rc\sigma^2) = -1$$

$$v = \frac{1}{1+rc\sigma^2}$$

- Thus we have the optimal linear incentive scheme.

- $v \downarrow$ in r
  $v \downarrow$ in c
  $v \downarrow$ in σ^2 } in general, $v = \frac{dq/da}{1+rc\sigma^2}$

$\Rightarrow v \uparrow$ in $\frac{dq}{da}$ (utility to principal

FB a^* can be arbitrarily closely approximated

$q < \underline{q} \Rightarrow k$ very low

if $\geq \underline{q} \rightarrow w^*$

$$\underbrace{l}_{\text{violates IC}} = \int_{-\infty}^{\underline{q}} [u(w^*(q)) - u(k)] f(q^*) dq$$

violates IC by this amount.

Can choose k and $\underline{q}$ to make l arbitrarily small.

Given $-M$, $\exists \underline{q}$ s.t. $\frac{f_a(q,a)}{f} \leq -M \quad q \leq \underline{q}$

$$\Rightarrow \frac{f_a}{f}\left(-\frac{1}{M}\right) \geq 1 \Leftrightarrow f \leq f_a\left(-\frac{1}{M}\right)$$

$$\Rightarrow l \leq \int_{-\infty}^{\underline{q}} (u(w^*(q)) - u(k)) \, f_a(q^*, q)\left(-\frac{1}{N}\right) dq$$

$$= O\left(-\frac{1}{M}\right)$$